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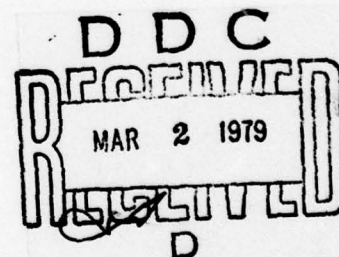
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Parametrically Excited Nonlinear Multi-Degree-  
Of-Freedom Systems

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scales, a perturbation technique, is used to develop solvability conditions for the various combinations of internal and parametric resonances considered.

Regions where trivial and nontrivial solutions exist are defined and the stability of the solutions within each region is discussed. Non trivial, unstable solutions have been shown to exist in regions where nontrivial stable solutions are known. Numerical solutions do not hint at the existence of these solutions.

The role of internal resonance in parametrically excited systems is explored. Strong modal interaction is demonstrated as a consequence of the presence of the cubic nonlinearity and the internal resonance. Because of this modal coupling, modes other than the one excited can dominate the response. A multiplicity of jumps is shown to exist.

Parametric excitation in the nonlinear flutter problem has been examined in detail for the first time. The effect of the parametric excitation can raise the flutter speed if the amplitude is small and lower it if the amplitude is large.

Limit-cycle behavior in the flutter problem is developed in a unique analytic way. The condition which predicts the onset of flutter is developed by using a linear analysis. The solution grows without bound. Interestingly, the same condition in the nonlinear analysis predicts the existence of a real nontrivial solution with a finite amplitude, the so-called limit-cycle.

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PARAMETRICALLY EXCITED NONLINEAR  
MULTIDEGREE-OF-FREEDOM SYSTEMS

by

Edward G. Tezak

Dissertation submitted to the Graduate Faculty of the  
Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Engineering Mechanics

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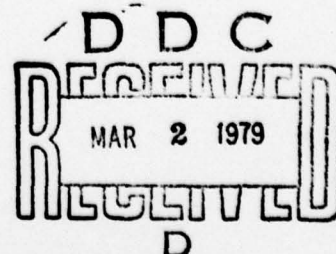
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## LIST OF SYMBOLS

- A = cross-section, also coefficient matrix, equation (9.9).
- $A_n$  = complex functions of  $T_1$  used in the perturbation scheme, equation (2.6).
- a = panel length.
- $a_n$  = twice the amplitude of  $A_n$ , equation (3.1).
- $\Delta a_n$  = perturbation of  $a_{ns}$  used to study the stability of the steady-state motion, equation (3.8).
- B = coefficient matrix, equation (9.9).
- C = damping coefficient.
- $C_n$  = modal damping coefficient.
- D = flexural rigidity.
- $D_n$  = partial differentiation with respect to  $T_n$ , equation (2.3d).
- E = elastic modulus.
- $E_n$  = terms in the solvability condition due to the parametric resonance, equation (2.9).
- F = amplitude of the parametric excitation, and coefficient matrix of parametric terms, equation (9.9) - (9.10).
- $f_{nm}$  = cross-modal amplitude of the excitation, equation (2.1).
- $G_{nm}$  = constants used to compute  $\Gamma_{nmpq}$ .
- h = panel thickness.
- I = moment of inertia of the cross-section area, identity matrix.
- $I_n$  = terms in the solvability condition due to the internal resonance, equation (2.9).
- Imag = imaginary part of a complex quantity.



# LIST OF SYMBOLS - Continued

- J = matrix of Jordan Form, equation (9.11).
- $K_n$  = terms in the solvability condition associated with the repeated frequency and the aerodynamic pressure, equation (5.36d).
- L = characteristic length used in the nondimensionalization scheme,  $zL = \ell$ .
- $\ell$  = dimensional length of the beam.
- M = Mach number.
- $P(t)$  = parametric excitation.
- $Q_n$  = coefficients in the solvability conditions for the clamped-hinged beam, equations (8.19) - (8.23).
- $r$  = radius of gyration of the cross-section area.
- $R_e$  = real part of a complex quantity.
- $T_0, T_1, T_2$  = fast and slow scales used in the perturbation scheme, equation (2.3a).
- $U_n(t; \epsilon)$  = time-dependent coefficients appearing in the expansion of  $w$ , equation (2.1) - (2.2).
- $U_{n_0}, U_{n_1}$  = functions of  $T_0$  and  $T_1$  appearing in the expansion of  $U_n$ , equation (2.2).
- $U_\infty$  = free stream velocity.
- $V$  = time-dependent coefficients in similarity transformation, equation (9.13).
- $w$  = the deflection.
- $Z$  = matrix used for similarity transformation, equation (9.13).

# SYMBOLS BASED ON THE GREEK ALPHABET

- $\alpha_n$  = phase of  $A_n$ , equation (3.1).
- $\alpha_{nm}$  = elements of aerodynamic pressure matrix, equation (9.8b).
- $\beta, \beta_i$  = phases introduced to put the solvability conditions in autonomous form, equations (4.18) - (4.21) and (8.27) - (8.35).
- $\beta_{nm}$  = coefficients of dynamic pressure terms, equation (5.1).
- $\Gamma_{nmpq}$  = coefficients appearing in the governing equations for  $U_n$ , equation (2.1).
- $\gamma_{nm}$  = coefficients appearing in the solvability condition, equation (2.9).
- $\Delta_{kn}$  = Kronecker delta.
- $\delta_n$  = scaling exponents of  $\epsilon$  used in nondistinct frequency analysis, equations (5.1) - (5.4).
- $\epsilon$  = small parameter used in the perturbation scheme, the square of the ratio of  $r$  to  $L$ .
- $\Lambda$  = aerodynamic detuning parameter, used as a measure of dynamic pressure equation (5.1).
- $\lambda$  = frequency of the parametric excitation, equation (2.1).
- $\tilde{\lambda}$  = stability parameter, used as a measure of dynamic pressure, equation (9.3).
- $\tilde{\lambda}_c$  = critical value of stability at onset of flutter.
- $\mu$  = convenient phase angle introduced to put the solvability conditions in autonomous form, equation (3.6c).
- $\mu_n$  = phases introduced to put the solvability conditions in autonomous form, equations (8.19) - (8.23).

# SYMBOLS BASED ON THE GREEK ALPHABET - Continued

- $\mu_s$  = steady-state value of  $\mu$ , equation (8.13).
- $\Delta\mu$  = perturbation of  $\mu$  used to study the stability of the steady-state motion, equation (3.8d).
- $\pi$  = 3.141...
- $\rho$  = detuning parameters for the external (parametric) resonances, equation (3.5a).
- $\rho_m, \rho_\infty$  = panel and air density.
- $\sigma$  = detuning parameter for the internal resonances, equations (4.1)-(4.4).
- $\phi_m$  = linear free-oscillation modes used to describe  $w$ .
- $\psi_n$  = normalizing factor for the free-oscillation modes.
- $\Omega$  = eigenvalues in the stability analysis.
- $\omega_m$  = natural frequencies (i.e., the eigenvalues corresponding to  $\phi_m$ ).



## CHAPTER ONE

### INTRODUCTION

Parametrically excited multi-degree-of-freedom systems have received considerable attention. Linear models of such systems predict regions of instability in the parameter space (in this case the parameters are essentially the frequency and the amplitude of the excitation) where the amplitude of the response grows without bound. The inclusion of damping in the model does not change the basic character of the results.

Of course, amplitudes of real systems do not grow without bound; the amplitudes are limited by nonlinear effects. Moreover, nonlinear effects make it possible for a bounded resonant motion to exist in regions where the linear models predict such motions are impossible. The interaction of the phase and amplitude, which is a characteristic of nonlinear systems, is responsible for this.

The system of governing equations is a set of second-order nonlinear ordinary differential equations having variable coefficients. The nonlinearity considered in this work is cubic in nature and considered small. Thus the equations may be referred to as weakly nonlinear. The excitation is considered to be harmonic. With some types of parametric resonance several modes can be strongly excited by a single harmonic frequency. Internal resonances can be responsible for strong modal coupling and, as a consequence, for a significant transfer of energy from one mode to another.



In what follows a general method of analyzing parametrically excited nonlinear multi-degree-of-freedom systems will be described in detail and applied to several physical examples. The parametric excitation is assumed to be harmonic with constant amplitude and frequency. Both internal and parametric resonances will be considered. The resonances to be examined include:

a) parametric

$$\lambda \approx \sum n_i \omega_i$$

b) internal

$$\omega_n = \sum a_i \omega_i$$

where the  $n$ ,  $n_i$ , and  $a_i$  are integers,  $\lambda$  represents the frequency associated with the excitation, and the  $\omega_i$ ,  $i = 1, 2, 3, \dots$ , represent the natural frequencies of the system. Modal damping is also included.

The response of the system is examined for the case where the natural frequencies,  $\omega_i$ , are distinct as well as for the case where there is one repeated frequency, i.e. the so-called flutter case. Regions describing the existence of trivial and nontrivial solutions are discussed along with the stability of these solutions within these regions. Before proceeding further a survey of the literature concerning parametrically excited nonlinear systems is in order.

### 1.1 Literature Review

A survey of the literature reveals that there is an abundance of material written on parametric, nonlinear and resonance phenomenon. A totally comprehensive review would be prohibitive here both in time and

space. The most complete effort in this direction has been accomplished by Nayfeh and Mook (1979).

Faraday (1831) appears to be the first to have observed the phenomenon of parametric resonance in conjunction with surface waves in fluid-filled cylinders. He was followed by Melde (1859), Strutt (1887), Stephenson (1906) and Raman (1912) who worked with vibrating strings. Stephenson (1908) showed that periodic loading of columns can have a stabilizing effect while Beliaev (1924) showed that a lateral motion can occur even though the axial loading of a column is below the static buckling load. Other early works on columns include Andronov and Leontovich (1927), Krylov and Bogoliubov (1935) and Chelomei (1939). Numerous books on parametric excitations include Bondarenko (1936), McLachlan (1947,1950), Den Hartog (1947), Minorsky (1947,1962), Stoker (1950), Hayashi, (1953a, 1964), Coddington and Levinson (1955), Malkin (1956), Cunningham (1958), Kauderer (1958), Bogoliubov and Mitropolsky (1961), Bolotin (1964), Andronov, Vitt, and Khaikin (1966), Meirovitch (1970), Cesari (1971), Nayfeh (1973) and Evan-Iwanowski (1976).

Early studies of nonlinear vibrations of bars were conducted by Woinowsky-Krieger (1950) and Burgreen (1951) and involved the use of elliptic functions in conjunction with an assumed single mode spatial function. Others who investigated the free oscillations of beams with hinged ends using various techniques include Wagner (1965), Srinivasan (1965,1967), Woodall (1966), Evensen (1968), Rehfield (1973, 1975) and Lou and Sikarskie (1925). Multiple theoretical and experimental investigations were conducted by Easley (1964b), Morris (1965), Srinivasan

(1966a), Bennett and Rinkel (1972), Bert and Fisher (1972) and Rehfield (1974a), while multi-mode forced responses were investigated by Eisley and Bennett (1970), Bennett and Eisley (1970) and Busby and Weingarten (1972). Straight and buckled beams were investigated by Tseng and Dugundji (1970, 1971) using harmonic balance. Others who analyzed buckled beams include Eisley (1964a), and Min and Eisley (1972).

Atluri (1973) examined beams with large curvatures and Nayfeh (1973a) studied a beam with slowly varying properties along its length. Both used the method of multiple scales. Nayfeh, Mook and Lobitz (1974) and Verma and Murthy (1974) analyzed nonuniform beams while Raju, Venkateswara and Kanaka Raju (1976) studied tapered beams.

A comprehensive treatment of the vibrations of linear flat plates is given in Leissa (1969). Nonlinear vibrations of plates has received extensive treatment. Yamaki (1961), Smith, Malme and Gogos (1961), Eisley (1964b), Murthy and Sherbourne (1972), Bayles, Lowery and Boyd (1973), Vendham (1975a,b), Crawford and Atluri (1975), Venkateswara, Raju and Kanaka Raju (1976a) all analyzed nonlinear vibrations of rectangular plates. Anisotropic rectangular plates were treated by Sathyamoorthy and Pandalai (1970), Bennett (1971) Chandra and Basava Raju (1975a,b) and Chandra (1976). Plates of a variety of shapes were investigated by numerous authors: circular by Yamaki (1961), Nowinski (1962), Bulkeley (1963) Huang and Sandman (1971), Huang (1972a,b, 1973, 1974) and Sridhar, Mook and Nayfeh (1975, 1978); annular by Sandman and Huang (1971), Huang and Woo (1973) and Huang, Woo and Walker (1976); triangular by Vendhan and Dhoopar (1973), Vendhan and Das (1975) and Vendhan, (1975b); and elliptic by Lobitz, Nayfeh and Mook (1977).



The phenomenon of flutter in plates and shells has received wide attention. The book by Dowell (1975) discusses the problem very succinctly. Other work in this area includes the study of flat plates by Kobayashi (1962a,b), Bolotin (1963), Dugundji (1966), Dowell (1966, 1967a, 1973), Morino (1969), Vetres and Dowell (1970), Eastep and McIntosh (1971), Kuo, Morino and Dugundji (1972) and Smith and Morino (1976). Numerous authors have studied curved plates and shells. Interaction of panel flutter with parametric excitation was analyzed by Dowell (1970a), Dzygadło (1970) and Kuo, Morino and Dugundji (1973).

## 1.2 Assessment of Previous Work

As demonstrated in the last section there is a tremendous volume of material which has been written on free and forced vibration; parametric excitations; linear and nonlinear systems; resonance; and single, two-degree and multi-degree-of-freedom systems. An attempt to comment on the value and limitations of each of these works and how each relates to the current discussion would be of marginal utility. Instead, the discussion will be limited to those few recent works which are the most relevant and which are significant contributions in themselves. Most authors cited here have discussed the various aspects of the analysis described in the introduction. However, no one has attempted to unify the analysis and make it comprehensive until now.

For instance, Tseng and Dugundji (1970, 1971) discussed straight and buckled beams using harmonic balance, but their analysis did not involve modal coupling. Atluri (1973) studied nonlinear vibrations of hinged beams for large curvatures but did not consider axial excitation.



Nayfeh (1973a) considered a beam with slowly varying properties along its length but did not consider modal coupling. Yamamoto and Saito (1970) analyzed parametrically excited multi-degree-of-freedom systems using the method of averaging. However, the system considered was linear and did not include repeated frequencies. Hsu (1963, 1965) also studied parametric excitation in multi-degree-of-freedom systems and developed stability criteria. He used the method of averaging and considered distinct frequencies. Tso and Asmis (1974) studied multiple parametric resonances in nonlinear systems. They discussed parametric resonances and combination resonances but confined their analysis to two-degree-of-freedom systems. In addition, they did not include an all important cubic term in the analysis. Fu and Nemat-Nasser (1972,1975) studied the stability of multi-degree-of-freedom systems and included the effect of repeated frequencies. However, their analysis was confined to linear systems. Another excellent work on parametrically excited linear systems having many degrees of freedom was conducted by Nayfeh and Mook (1977). Additional studies were conducted by Sugiyama, Fujiwara, and Sekiya (1968) using analog simulation, and they found combination resonances. Iwatsubo, Saigo and Sugiyama (1973) and Iwatsubo, Sugiyama and Ogino (1974) performed theoretical and experimental investigations on parametrically excited columns and also demonstrated the existence of combination resonances. They included the effects of internal and external damping in their analysis but not internal resonance. The effects of internal resonance have been virtually ignored in most works until recently. The importance of internal resonance in the analysis of

ship motions has been demonstrated by Nayfeh, Mook and Marshall (1973, 1974) and Mook, Marshall and Nayfeh (1974). Its importance in the analysis of structural vibrations has been shown by Nayfeh, Mook and Sridhar (1973), Nayfeh, Mook and Lobitz (1974), Sridhar, Nayfeh and Mook (1975) and Mook, Sridhar and Nayfeh (1978). These latter works have been a major advancement toward unifying nonlinear, resonance studies. However, they do not include parametric excitations.

The majority of work discussed so far deals with distinct frequencies. When the natural frequencies are no longer distinct the response enters the so-called flutter mode or condition. The book by Dowell (1975) provides an insight into the mechanism that can cause flutter. Morino (1969) and Smith and Morino (1976) have presented an excellent discussion demonstrating limit cycles and stability. However, neither included the effect of an in-plane excitation. The role of a parametric excitation has thus been neglected. The numerical works of Dowell (1966, 1970) suffer from the same deficiency as those of Kuo, Morino and Dugundji (1972). Constant in-plane loads were used in these analyses. A harmonic excitation and its influence on flutter were considered by Dowell (1974), Dzygadło (1970) and Kuo, Morino and Dugundji (1973). However, the excitation was not parametric in nature.

### 1.3 Contributions of the Current Work

Though a vast amount of work has been cited here, there are some significant facets of analysis of parametrically excited multi-degree-of-freedom systems that have not been examined. The role that internal resonance plays has been virtually ignored. Parametric excitations

in the nonlinear flutter problem has been similarly neglected. In addition, there has been no single method of analysis that is general in nature and applicable to a wide variety of problems. This work investigates these fascinating aspects of multi-degree-of-freedom systems and thus fills a longstanding void. In the process a unified versatile approach is developed and some interesting behavior is discussed. The following contributions are unique and significant:

a. A general method for analyzing parametrically excited nonlinear multi-degree-of-freedom systems is developed. To demonstrate its versatility, we apply the method to the following physical systems:

- 1) Systems with distinct natural frequencies
  - a) without internal resonance
  - b) with internal resonance
- 2) Systems with repeated natural frequencies
  - a) without internal resonance
  - b) with internal resonance

b. Regions where trivial and nontrivial solutions exist are defined and the stability of the solutions within each region is discussed. Nontrivial, unstable solutions have been shown to exist in regions where nontrivial stable solutions are known. Numerical solutions do not hint at the existence of these solutions.

c. The role of internal resonance in parametrically excited, nonlinear systems is explored. Strong modal interaction is demonstrated as a consequence of the presence of the cubic nonlinearity and the internal resonance. Because of this modal coupling, modes other than



the one excited can dominate the response. A multiplicity of jumps is shown to exist.

d. Parametric excitation in the nonlinear flutter problem has been examined in detail for the first time, and includes the effects of internal resonance.

e. Limit-cycle behavior in the flutter problem is developed in a unique analytic way. The condition which predicts the onset of flutter is developed by using a linear analysis. The solution grows without bound. Interestingly, the same condition in the nonlinear analysis predicts the existence of a real nontrivial solution with a finite amplitude, the so-called limit cycle.

f. The current work is divided into several parts. In Part I, the analysis deals with distinct natural frequencies. Chapter 2 is devoted to the problem formulation and generation of the solvability conditions. Chapter 3 deals with the analysis of parametric resonance in the absence of internal resonance. Chapter 4 is the heart of the analysis and includes the internal as well as the parametric resonance.

In Part II, the same general approach is followed for the analysis of non-distinct natural frequencies. Chapter 5 develops the limit-cycle behavior and the general solvability conditions for the situation involving repeated frequencies. Chapter 6 considers parametric resonance in the absence of internal resonance and Chapter 7 discusses internal and parametric resonance.

Part III is devoted to numerical examples. Chapter 8 deals with the case of distinct frequencies and uses the beam as an example. Chapter 9 develops the flutter problem for a simply supported plate.

## CHAPTER TWO

### PROBLEM FORMULATION AND METHOD OF SOLUTION

In this chapter the nonlinear system studied is defined. Modification of the basic system is made in a subsequent chapter when the case of non-distinct natural frequencies is discussed. Solvability conditions are developed by using the method of multiple scales.

#### 2.1 Problem Formulation

The system to be analyzed is governed by the following set of ordinary-differential equations:

$$\ddot{U}_n + \omega_n^2 U_n = \epsilon \left[ -2 \cos \lambda t \sum_{m=1}^{\infty} f_{nm} U_m - 2 C_n \dot{U}_n + \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \Gamma_{nmpq} U_m U_p U_q \right]$$

for  $n = 1, 2, \dots$  (2.1)

where the  $\omega_n$  are the distinct natural frequencies corresponding to the linear free-oscillation modes;  $\epsilon$  is a small dimensionless parameter; the  $f_{nm}$  are the amplitudes associated with the harmonic parametric excitation;  $\lambda$  is the constant frequency of excitation; the  $C_n$  are the modal viscous damping coefficients; and the  $\Gamma_{nmpq}$  are constant coefficients.

Experience has shown that a straightforward perturbation expansion for  $U_n$ , for small  $\epsilon$ , results in the emergence of secular terms, which make this expansion not uniformly valid for large  $t$ . Hence, in order to obtain a uniformly valid expansion, we employ the method of multiple scales. Specifically, the derivative-expansion version of the method of multiple scales is utilized to study the steady-state solutions and their stability.

## 2.2 Method of Solution

Following the method of multiple scales (Nayfeh, 1973), we assume expansions for the  $U_n$  in the form

$$U_n(t; \epsilon) = U_{n0}(T_0, T_1) + \epsilon U_{n1}(T_0, T_1) + \dots \quad (2.2)$$

where

$$T_0 = t \text{ and } T_1 = \epsilon t \quad (2.3a)$$

The derivatives become

$$\frac{d}{dt} = D_0 + \epsilon D_1 + O(\epsilon^2) \quad (2.3b)$$

and

$$\frac{d^2}{dt^2} = D_0^2 + \epsilon 2D_0 D_1 + O(\epsilon^2) \quad (2.3c)$$

where

$$D_0 = \frac{\partial}{\partial T_0} \text{ and } D_1 = \frac{\partial}{\partial T_1} \quad (2.3d)$$

Substituting equations (2.2) and (2.3) into equation (2.1) and equating the coefficients of  $\epsilon^0$  and  $\epsilon$ , we obtain

$$D_0^2 U_{n0} + \omega_n^2 U_{n0} = 0 \quad (2.4)$$

$$\begin{aligned} D_0^2 U_{n1} + \omega_n^2 U_{n1} = & -2D_0 D_1 U_{n0} - 2\cos\lambda T_0 \sum_{m=1}^{\infty} f_{nm} U_m - 2C_n D_0 U_{n0} \\ & + \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \Gamma_{nmpq} U_{m0} U_{p0} U_{q0} \text{ for } n = 1, 2, \dots \end{aligned} \quad (2.5)$$

We can write the solution of equations (2.4) in the form

$$U_{n0} = A_n(T_1) \exp(i\omega_n T_0) + cc \quad (2.6)$$

for  $n = 1, 2, \dots$ , where  $cc$  stands for the complex conjugate of the preceding term and the  $A_n$  are, at this point, unspecified complex function



of  $T_1$ . They are determined by eliminating "troublesome" terms from the  $U_{n1}$ .

Substituting equation (2.6) into equation (2.5) leads to

$$\begin{aligned}
 D_0^2 U_{n1} + \omega_n^2 U_{n1} = & -2i\omega_n(D_1 A_n + C_n A_n) \exp(i\omega_n T_0) - \sum_{m=1}^{\infty} f_{nm} A_m \{ \exp[i(\omega_m \\
 & + \lambda)T_0] + \exp[i(\omega_m - \lambda)T_0] \} + \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \Gamma_{nmpq} \{ A_m A_p A_q \times \\
 & \exp[i(\omega_m + \omega_p + \omega_q)T_0] + A_m A_p \bar{A}_q \exp[i(\omega_m + \omega_p - \omega_q)T_0] \\
 & + A_m \bar{A}_p A_q \exp[i(\omega_m - \omega_p + \omega_q)T_0] + A_m \bar{A}_p \bar{A}_q \exp[i(\omega_m - \omega_p - \omega_q)T_0] \} \\
 & + cc
 \end{aligned} \tag{2.7}$$

where the overbars denote the complex conjugates.

The terms containing the factor  $\exp(\pm i\omega_n T_0)$  in equation (2.7) lead to so-called secular terms in  $U_{n1}$ , while terms containing factors such as  $\exp(i\Omega T_0)$ , where  $\Omega - \omega_n = O(\epsilon)$ , lead to so-called small-divisor terms in  $U_{n1}$ . The former renders the expansion nonuniform ( $U_{n1}/U_{n0}$  is not bounded) as  $t$  increases, while the latter are inconsistent with the assumed expansion (2.2) because they raise  $\epsilon U_{n1}$  to the same order as  $U_{n0}$ . Consequently, both kinds of terms are "troublesome" and must be eliminated for a uniform expansion.

Secular terms, for example, are associated with the combination

$$\omega_m + \omega_p - \omega_q$$

when  $m = n$  and  $p = q$ . In addition to all these possible combinations, there may be small-divisor terms associated with combinations having the form

$$\omega_n = a\omega_m + b\omega_p + c\omega_q + \epsilon\sigma_n \tag{2.8a}$$

where  $a$ ,  $b$ , and  $c$  are integers such that  $|a| + |b| + |c| = 3$ , and  $\sigma_n$  is a detuning parameter. If equation (2.8a) is satisfied, an internal resonance is said to exist. Finally, if there are combinations having the form

$$\lambda = a\omega_m + b\omega_n + \varepsilon\rho \quad (2.8b)$$

where  $a$  and  $b$  are integers such that  $|a| + |b| = 2$ , and  $\rho$  is another detuning parameter, additional small-divisor terms arise. If equation (2.8b) is satisfied, a parametric resonance is said to exist.

It follows that the general form of the conditions that eliminate the troublesome terms from the  $U_{n1}$  is

$$-2i\omega_n(D_1A_n + C_nA_n) + A_n \sum_{p=1}^{\infty} \gamma_{np}A_p\bar{A}_p + I_n + E_n = 0 \text{ for } n = 1, 2, \dots \quad (2.9)$$

where

$$\gamma_{np} = \begin{cases} 3\Gamma_{nnnn} & \text{if } p = n \\ 2(\Gamma_{nnpp} + \Gamma_{npnp} + \Gamma_{nppn}) & \text{if } p \neq n \end{cases}$$

The terms  $I_n(A_1, A_2, \dots)$  and  $E_n(A_1, A_2, \dots)$  are complex functions, resulting from the internal and parametric resonances, respectively, if any exist.

### 2.3 Summary

A general approach to the solution of equations (2.1) has been developed using the method of multiple scales to obtain a uniformly valid solution. The solvability conditions yielded equations (2.9),

which are general and can be applied to a large variety of problems, depending on the form of the internal and parametric resonances involved.



## CHAPTER THREE

### PARAMETRIC RESONANCE IN THE ABSENCE OF INTERNAL RESONANCE

In this chapter parametric resonance is examined in detail. The effect of an internal resonance is discussed in the next chapter. Various cases are considered and some general observations are given. The emphasis is on the steady-state response and its stability.

#### 3.1 No Internal Resonance

In this case  $I_n \equiv 0$  for all modes and  $E_n = 0$  for all modes that are not part of any resonance with the excitation. The  $E_n$  for the resonating modes depend on the resonant combinations of frequencies. In the first case considered, there is neither internal nor parametric resonance.

#### 3.2 The Case of Modes Not Involved in a Parametric Resonance

For modes not associated with a parametric or internal resonance  $I_n = E_n \equiv 0$  in (2.9). Now we let

$$A_n = \frac{1}{2} a_n(T_1) \exp[i\alpha_n(T_1)] \quad (3.1)$$

where the amplitude  $a_n$  and the phase  $\alpha_n$  are real functions of  $T_1$ . Upon substituting (3.1) into the solvability condition (2.9) and separating the resulting equation into its real and imaginary parts, we obtain

$$\omega_n(a_n' + C_n a_n) = 0 \quad (3.2a)$$

$$\omega_n a_n \alpha_n' + \frac{1}{8} a_n \sum_p \gamma_{np} a_p^2 = 0 \quad (3.2b)$$

where the primes designate differentiation with respect to  $T_1$ . The solution of (3.2a) is

$$a_n = a_0 \exp[-C_n T_1] \quad (3.3)$$

and it follows that  $a_n \rightarrow 0$  as  $T_1 \rightarrow \infty$ ; so that the steady-state amplitude  $a_n = 0$ . Therefore, the steady-state solution has the form

$$U_n = 0(\epsilon) \quad (3.4)$$

Consequently, only the directly excited modes and the modes involved in an internal resonance can be part of the first approximation of the steady-state response.

### 3.3 The Case Where $\lambda \approx 2\omega_k$ , $n \neq k$

The first case to be considered is the case of a parametric resonance where  $\lambda \approx 2\omega_k$ . A detuning parameter,  $\rho$ , is introduced and used to express the nearness of  $\lambda$  to  $2\omega_k$  quantitatively according to

$$\lambda = 2\omega_k + \epsilon\rho \quad (3.5a)$$

It follows that

$$(\lambda - \omega_k)T_0 = \omega_k T_0 + \epsilon T_0 \rho = \omega_k T_0 + \rho T_1 \quad (3.5b)$$

Then for  $n \neq k$  the resonant terms become

$$E_n = 0 \quad \text{for } n \neq k \quad (3.5c)$$

and

$$E_k = -f_{kk} \bar{A}_k \exp(i\rho T_1) = -\frac{f_{kk}}{2} a_k \exp[i(\rho T_1 - \alpha_k)] \quad (3.5d)$$

Substituting (3.5) into the solvability conditions (2.9) and separating into real and imaginary parts, we obtain

$$\omega_k(a'_k + c_k a_k) + \frac{f_{kk}}{2} a_k \sin \mu = 0 \quad (3.6a)$$

and

$$\omega_k a_k \alpha'_k + \frac{1}{8} a_k \sum_{p=1}^{\infty} \gamma_{kp} a_p^2 - \frac{f_{kk}}{2} a_k \cos \mu = 0 \quad (3.6b)$$

where

$$\mu = \rho T_1 - 2\alpha_k \quad (3.6c)$$

For the steady-state solution all the  $a'_n = 0$  and  $\mu' = 0$ . From equations (3.3) and (3.4) it follows that  $a_n \rightarrow 0$  as  $T_1 \rightarrow \infty$  so that the steady-state amplitude  $a_n = 0$  for  $n \neq k$ . The trivial solution  $a_k = 0$  is a solution of equations (3.6). For nontrivial solutions  $a_k \neq 0$ , equations (3.6) reduce to

$$\omega_k c_k + \frac{f_{kk}}{2} \sin \mu = 0 \quad (3.7a)$$

and

$$\frac{\rho \omega_k}{2} + \frac{1}{8} \gamma_{kk} a_k^2 - \frac{f_{kk}}{2} \cos \mu = 0 \quad (3.7b)$$

where

$$\mu' = 0 = \rho - 2\alpha'_k \quad (3.7c)$$

Equations (3.7) may now be solved for  $a_k$  and  $\mu$ . Squaring and adding (3.7a) and (3.7b) leads to

$$\omega_k^2 c_k^2 + \left( \frac{\rho \omega_k}{2} + \frac{1}{8} \gamma_{kk} a_k^2 \right)^2 = \frac{f_{kk}^2}{4}$$

or

$$a_k = \sqrt{\frac{8\omega_k}{\gamma_{kk}} \left[ -\frac{\rho}{2} \pm \sqrt{\frac{f_{kk}^2}{4\omega_k^2} - c_k^2} \right]} \quad (3.7d)$$



We note that for nontrivial solutions to exist

$$|f_{kk}| \geq 2\omega_k C_k \quad (3.7e)$$

when  $\rho > 0$  and  $\gamma_{kk} < 0$ . When  $\rho < 0$  and  $\gamma_{kk} < 0$

$$\rho^2 < \frac{f_{kk}^2}{\omega_k^2} - 4C_k^2 \quad (3.7f)$$

is required for nontrivial solutions. A more complete discussion of the regions where solutions exist is contained in the numerical examples in Chapter 8. We note here that  $\gamma_{kk} < 0$  in those examples.

When equation (3.7d) is substituted into equations (2.2) and (2.6), the resulting steady-state solution becomes

$$U_n = 0(\epsilon) \quad n \neq k \quad (3.7g)$$

and

$$U_k = a_k \cos[(\omega_k + \epsilon \frac{\rho}{2})t + \tau_k] + 0(\epsilon) \quad (3.7h)$$

where the phase  $\tau_k$  is a constant that depends on the initial conditions.

The stability of the steady-state solution can now be studied by determining the behavior of the system when it is perturbed slightly.

Thus, we let

$$a_k = \hat{a}_k + \Delta a_k \quad (3.8a)$$

$$\mu = \hat{\mu} + \Delta \mu \quad (3.8b)$$

where  $\hat{a}_k$  and  $\hat{\mu}$  represent steady-state values. Substituting (3.8) into (3.6) yields

$$2\omega_k(\hat{a}_k' + \Delta a_k' + C_k \hat{a}_k + C_k \Delta a_k) + f_{kk}(\hat{a}_k + \Delta a_k) \sin(\hat{\mu} + \Delta \mu) = 0 \quad (3.9a)$$

and

$$\omega_k (\hat{a}_k + \Delta a_k) \alpha'_k + \frac{1}{8} \gamma_{kk} (a_k + \Delta a_k)^3 - \frac{f_{kk}}{2} (\hat{a}_k + \Delta a_k) \cos(\hat{\mu} + \Delta \mu) = 0 \quad (3.9b)$$

After expanding, neglecting products of small terms, and noting that  $\hat{a}'_k = 0$ , we obtain

$$2\omega_k (\Delta a'_k + C_k \Delta a_k) + 2\omega_k \hat{a}_k C_k + f_{kk} \Delta a_k \sin \hat{\mu} + f_{kk} \hat{a}_k \sin \hat{\mu} + f_{kk} \hat{a}_k \cos \hat{\mu} \Delta \mu = 0 \quad (3.9c)$$

$$\omega_k a_k \left( \frac{\rho - \Delta \mu}{2} \right) + \omega_k \Delta a_k \left( \frac{\rho}{2} \right) + \frac{1}{8} \gamma_{kk} \hat{a}_k^3 + \frac{3}{8} \gamma_{kk} \hat{a}_k^2 \Delta a_k - \frac{f_{kk}}{2} \hat{a}_k \cos \hat{\mu} + \frac{f_{kk}}{2} \hat{a}_k \sin \hat{\mu} \Delta \mu - \frac{f_{kk}}{2} \Delta a_k \cos \hat{\mu} = 0 \quad (3.9d)$$

Recalling the steady-state equations and letting

$$(\Delta a_k, \Delta \mu) \propto \exp(\Omega T_1)$$

we obtain

$$(2\omega_k \Omega) \Delta a_k + (f_{kk} \hat{a}_k \cos \hat{\mu}) \Delta \mu = 0 \quad (3.10a)$$

$$- \left( \frac{1}{4} \gamma_{kk} \hat{a}_k \right) \Delta a_k + \left( \frac{\omega_k \Omega}{2} + \omega_k C_k \right) \Delta \mu = 0 \quad (3.10b)$$

from which

$$\omega_k^2 \Omega^2 + 2\omega_k^2 C_k \Omega + \frac{1}{4} \gamma_{kk} \hat{a}_k^2 f_{kk} \cos \hat{\mu} = 0 \quad (3.10c)$$

or

$$\Omega^2 + 2C_k \Omega + \frac{\gamma_{kk} f_{kk} \hat{a}_k^2}{4\omega_k^2} \cos \hat{\mu} = 0 \quad (3.10d)$$

which has the solution

$$\Omega = -C_k \pm \sqrt{C_k^2 - \frac{\gamma_{kk} f_{kk} \hat{a}_k^2}{4\omega_k^2} \cos \hat{\mu}} \quad (3.11)$$

For stable solutions the real part of  $\Omega$  must be negative, or zero i.e.,

$$\text{Re}(\Omega) \leq 0 \quad (3.12)$$

or for stability

$$\cos \hat{\mu} \geq 0 \quad (3.13)$$

Hence, the  $\rho$  for stability can be determined from equation (3.7d).

Additional discussion of the ramifications of the stability conditions will be presented in Chapter 8 in the numerical examples.

### 3.4 The Case Where $\lambda \approx \omega_m + \omega_k$ , $m \neq k$

The case of parametric resonance where  $\lambda \approx \omega_m + \omega_k$  is analyzed.

The detuning parameter  $\rho$  is now used to measure the nearness of  $\lambda$  to  $\omega_m + \omega_k$  according to:

$$\lambda = \omega_m + \omega_k + \epsilon \rho \quad (3.14a)$$

It follows that

$$(\lambda - \omega_m)T_0 = \omega_k T_0 + \epsilon T_0 \rho = \omega_k T_0 + \rho T_1 \quad (3.14b)$$

and

$$(\lambda - \omega_k)T_0 = \omega_m T_0 + \epsilon T_0 \rho = \omega_m T_0 + \rho T_1 \quad (3.14c)$$

$$E_n = 0 \quad \text{for } n \neq k \text{ or } m \quad (3.15a)$$

$$E_m \exp(i\omega_m T_0) = -f_{mk} \bar{A}_k \exp[i(\lambda - \omega_k)T_0] = -f_{mk} \bar{A}_k \exp[i(\omega_m T_0 + \rho T_1)] \quad (3.15b)$$

and

$$E_k \exp(i\omega_k T_0) = -f_{km} \bar{A}_m \exp[i(\lambda - \omega_m)T_0] = -f_{km} \bar{A}_m \exp[i(\omega_k T_0 + \rho T_1)] \quad (3.15c)$$



Using the polar notation in equation (3.1), we can write

$$E_m = - \frac{f_{mk} a_k}{2} \exp[i(\rho T_1 - \alpha_k)] \quad (3.15d)$$

and

Substituting equations (3.15) into equation (2.9), dividing by  $\exp(i\alpha_n)$ , and separating the resulting equations into real and imaginary parts, we obtain

$$\omega_m (a'_m + C_m a_m) + \frac{f_{mk}}{2} a_k \sin \mu = 0 \quad (3.16a)$$

$$\omega_k (a'_k + C_k a_k) + \frac{f_{km}}{2} a_m \sin \mu = 0 \quad (3.16b)$$

$$\omega_m a_m \alpha'_m + \frac{1}{8} a_m \sum_{p=1}^{\infty} \gamma_{mp} a_p^2 - \frac{f_{mk}}{2} a_k \cos \mu = 0 \quad (3.16c)$$

$$\omega_k a_k \alpha'_k + \frac{1}{8} a_k \sum_{p=1}^{\infty} \gamma_{kp} a_p^2 - \frac{f_{km}}{2} a_m \cos \mu = 0 \quad (3.16d)$$

where

$$\mu = \rho T_1 - \alpha_k - \alpha_m \quad (3.16e)$$

For  $n \neq k$ , or  $m$ , the amplitudes and phases are governed by equations (3.2) and (3.3); thus, these modes do not appear in the first approximation. For the steady-state solution all  $a'_n = 0$  and  $\mu' = 0$ . It follows that  $a_k = a_m = 0$  is a solution to equation (3.16), but nontrivial solution (i.e.,  $a_k \neq 0$  and  $a_m \neq 0$ ) are desired. Hence, the steady-state version of equations (3.16a) and (3.16b) becomes

$$\omega_m C_m a_m + \frac{f_{mk}}{2} a_k \sin \mu = 0 \quad (3.17a)$$

$$\omega_k C_k a_k + \frac{f_{km}}{2} a_m \sin \mu = 0 \quad (3.17b)$$

Solving equations (3.16c,d) for  $\alpha'_m$  and  $\alpha'_k$ , respectively, and substituting the results into equation (3.16e), when differentiated, leads to

$$\rho + \frac{1}{8} \left[ \frac{\gamma_{mm}}{\omega_m} + \frac{\gamma_{km}}{\omega_k} \right] a_m^2 + \frac{1}{8} \left[ \frac{\gamma_{mk}}{\omega_m} + \frac{\gamma_{kk}}{\omega_k} \right] a_k^2 - \frac{1}{2} \left[ \frac{f_{mk} a_k}{a_m \omega_m} + \frac{f_{km} a_m}{a_k \omega_k} \right] \cos \mu = 0 \quad (3.18a)$$

Combining equations (3.17a) and (3.17b) leads to

$$a_m = \sqrt{\frac{\omega_k C_k f_{mk}}{\omega_m C_m f_{km}}} a_k \quad (3.18b)$$

Hence, for nontrivial solutions to exist  $f_{mk}$  and  $f_{km}$  must have the same sign. Substituting equation (3.18b) into equation (3.18a) and dividing by  $a_k^2$ , we obtain

$$\rho \left( \frac{\omega_k C_k f_{mk}}{\omega_m C_m f_{km}} \right)^{1/2} - \frac{1}{2} \frac{f_{mk}}{\omega_m} \left( 1 + \frac{C_k}{C_m} \right) \cos \mu + \frac{1}{8} \left( \frac{\omega_k C_k f_{mk}}{\omega_m C_m f_{km}} \right)^{1/2} \times$$

$$\left[ \left( \frac{\gamma_{mm}}{\omega_m} + \frac{\gamma_{km}}{\omega_k} \right) \left( \frac{\omega_k C_k f_{mk}}{\omega_m C_m f_{km}} \right) + \left( \frac{\gamma_{mk}}{\omega_m} + \frac{\gamma_{kk}}{\omega_k} \right) \right] a_k^2 = 0 \quad (3.18c)$$

and hence

$$a_k = \frac{\sqrt{\frac{1}{2} \frac{f_{mk}}{\omega_m} \left( 1 + \frac{C_k}{C_m} \right) \cos \mu - \rho \left( \frac{\omega_k C_k f_{mk}}{\omega_m C_m f_{km}} \right)^{1/2}}}{\sqrt{\frac{1}{8} \left( \frac{\omega_k C_k f_{mk}}{\omega_m C_m f_{km}} \right)^{1/2} \left[ \left( \frac{\gamma_{mm}}{\omega_m} + \frac{\gamma_{km}}{\omega_k} \right) \left( \frac{\omega_k C_k f_{mk}}{\omega_m C_m f_{km}} \right) + \frac{\gamma_{mk}}{\omega_m} + \frac{\gamma_{kk}}{\omega_k} \right]}} \quad (3.18d)$$

The steady state solution now has the following form:

$$U_n = O(\epsilon) \quad n \neq k \text{ or } m \quad (3.19a)$$

$$U_m = a_m \cos[(\omega_m + \epsilon \alpha'_m)t + \tau_m] + O(\epsilon) \quad (3.19b)$$

$$U_k = a_k \cos[(\omega_k + \epsilon \alpha'_k)t + \tau_k] + O(\epsilon) \quad (3.19c)$$

where  $\tau_m$  and  $\tau_k$  are constants depending on initial conditions. The nonlinearity adjusts the frequencies so that

$$\omega_m + \varepsilon \alpha_m' + \omega_k + \varepsilon \alpha_k' = \omega_m + \omega_k + \varepsilon \rho = \lambda \quad (3.20)$$

If both solutions ( $a_m = a_k = 0$  and  $a_m \neq 0, a_k \neq 0$ ) are stable, then the initial conditions determine which solution represents the response.

The stability of the system may be examined by the same method used in the case of  $\lambda$  near  $2\omega_k$ .

### 3.5 The Case Where $\lambda \approx \omega_k - \omega_m, m \neq k$

The case of parametric resonance where  $\lambda \approx \omega_k - \omega_m$  is analyzed. The detuning parameter,  $\rho$ , is introduced and used to express the nearness of  $\lambda$  to  $\omega_k - \omega_m$ , according to

$$\lambda = \omega_k - \omega_m + \varepsilon \rho \quad (3.21a)$$

• from which we can write

$$(\omega_k - \lambda)T_0 = (\omega_m - \varepsilon \rho)T_0 = \omega_m T_0 - \rho T_1 \quad (3.21b)$$

and

$$(\lambda + \omega_m)T_0 = (\omega_k + \varepsilon \rho)T_0 = \omega_k T_0 + \rho T_1 \quad (3.21c)$$

$$E_n = 0 \text{ for } n \neq k \text{ or } m \quad (3.22a)$$

$$E_k \exp[i\omega_k T_0] = -f_{km} A_m \exp[i(\lambda + \omega_m)T_0] = -f_{km} A_m \exp[i(\omega_k T_0 + \rho T_1)] \quad (3.22b)$$

and

$$E_m \exp[i\omega_m T_0] = -f_{mk} A_k \exp[i(\omega_m - \lambda)T_0] = -f_{mk} A_k \exp[i(\omega_m T_0 - \rho T_1)] \quad (3.22c)$$

Using the polar notation in equation (3.1), we can write

$$E_k = -\frac{f_{km} a_m}{2} \exp[i(\rho T_1 + \alpha_m)] \quad (3.22d)$$



and

$$E_m = - \frac{f_{mk} a_k}{2} \exp[i(-\rho T_1 + \alpha_k)] \quad (3.22e)$$

Substituting equation (3.22) into equation (2.9), dividing by  $\exp(i\alpha_k)$  and  $\exp(i\alpha_m)$ , respectively, and separating the resulting equations into real and imaginary parts leads to

$$\omega_m(a'_m + C_m a_m) - \frac{f_{mk} a_k}{2} \sin \mu = 0 \quad (3.23a)$$

$$\omega_k(a'_k + C_k a_k) + \frac{f_{km} a_m}{2} \sin \mu = 0 \quad (3.23b)$$

$$\omega_m a_m \alpha'_m + \frac{a_m}{8} \sum_{p=1}^{\infty} \gamma_{mp} a_p^2 - \frac{f_{mk} a_k}{2} \cos \mu = 0 \quad (3.23c)$$

$$\omega_k a_k \alpha'_k + \frac{a_k}{8} \sum_{p=1}^{\infty} \gamma_{kp} a_p^2 - \frac{f_{km} a_m}{2} \cos \mu = 0 \quad (3.23d)$$

where

$$\mu = \rho T_1 - \alpha_k + \alpha_m \quad (3.23e)$$

For  $n \neq k$  or  $m$ , the amplitudes and phases are governed by equations (3.2) and (3.3). For the steady-state solution all  $a'_n = 0$  and  $\mu' = 0$ . It follows that  $a_k = a_m = 0$  is a solution to equations (3.23), but non-trivial solutions (i.e.,  $a_k \neq 0$  and  $a_m \neq 0$ ) are desired. Hence, the steady-state version of equations (3.23) becomes

$$\omega_m C_m a_m - \frac{f_{mk} a_k}{2} \sin \mu = 0 \quad (3.24a)$$

$$\omega_k C_k a_k + \frac{f_{km} a_m}{2} \sin \mu = 0 \quad (3.24b)$$

and

$$\begin{aligned} \rho + \frac{1}{8} \left[ \frac{-\gamma_{mm}}{\omega_m} + \frac{\gamma_{km}}{\omega_k} \right] a_m^2 + \frac{1}{8} \left[ \frac{-\gamma_{mk}}{\omega_m} + \frac{\gamma_{kk}}{\omega_k} \right] a_k^2 \\ - \frac{1}{2} \left[ \frac{-f_{mk} a_k}{a_m \omega_m} + \frac{f_{km} a_m}{a_k \omega_k} \right] \cos \mu = 0 \end{aligned} \quad (3.24c)$$

which leads to

$$a_m = \sqrt{-\frac{\omega_k C_k f_{mk}}{\omega_m C_m f_{km}}} a_k \quad (3.25)$$

Thus, a solution can exist only if  $f_{km}$  and  $f_{mk}$  have the different signs. In this case, the result for  $a_k$  can be obtained from (3.18d) by simply changing the sign of  $\omega_m$ .

### 3.6 Summary

Various cases concerning a parametric resonance have been discussed. The following observations are pertinent in the absence of an internal resonance:

- (a) Only modes which are directly excited by a parametric resonance are part of the steady-state response.
- (b) For combination-type resonances the lower mode appears to be dominant in the response.
- (c) The steady-state response is essentially that of the forced linear-oscillation modes with the frequency being adjusted.
- (d) Stability criteria have been outlined when more than one stable solution exists, the initial conditions will determine the response.

More insight into the response in the presence of parametric resonance will be gained in Chapter 8 where specific numerical examples are presented.

## CHAPTER FOUR

### THE EFFECT OF INTERNAL RESONANCE

In this chapter we discuss the effect of an internal resonance. Specifically, we examine in detail the case where  $\omega_k \approx 3\omega_m$ . Part of the rationale for this choice is that this combination occurs in physical systems such as the beam considered in the numerical example in Chapter 8.

The parametric resonances of the last chapter are examined in the presence of this internal resonance. See Chapter 2 for a review of internal resonance.

An internal resonance emerges when equation (2.8a) is satisfied. When the form of equation (2.8a) is

$$\omega_k = \omega_m + \omega_p + \omega_q + \epsilon\sigma \quad (4.1)$$

it follows from equation (2.7) that the terms involved in the internal resonance are given by the following:

$$I_k = \hat{\Gamma}_k A_m A_p A_q \exp(-i\sigma T_1) \quad (4.2a)$$

$$I_m = \hat{\Gamma}_m A_k \bar{A}_p \bar{A}_q \exp(i\sigma T_1) \quad (4.2b)$$

$$I_p = \hat{\Gamma}_p A_k \bar{A}_m \bar{A}_q \exp(i\sigma T_1) \quad (4.2c)$$

$$I_q = \hat{\Gamma}_q A_k \bar{A}_m \bar{A}_p \exp(i\sigma T_1) \quad (4.2d)$$

$$\hat{\Gamma}_k = \Gamma_{kmpq} + \Gamma_{kpmq} + \Gamma_{kpqm} + \Gamma_{kqpm} + \Gamma_{kqmp} + \Gamma_{kmpq} \quad (4.3a)$$

$$\hat{\Gamma}_m = \Gamma_{mkpq} + \Gamma_{mkqp} + \Gamma_{mpkq} + \Gamma_{mqkp} + \Gamma_{mpqk} + \Gamma_{mpqk} \quad (4.3b)$$



$$\hat{\Gamma}_p = \Gamma_{pmqk} + \Gamma_{pqmk} + \Gamma_{pmkq} + \Gamma_{pqkm} + \Gamma_{pkmq} + \Gamma_{pkqm} \quad (4.3c)$$

$$\hat{\Gamma}_q = \Gamma_{qmpk} + \Gamma_{qpmk} + \Gamma_{qmkp} + \Gamma_{qpkm} + \Gamma_{qkmp} + \Gamma_{qkpm} \quad (4.3d)$$

When the form of equation (2.8a) is

$$\omega_k = \omega_m + 2\omega_p + \epsilon\sigma \quad (4.4)$$

it follows from equation (2.7) that

$$I_k = \hat{\Gamma}_k A_m A_p^2 \exp(-i\sigma T_1) \quad (4.5a)$$

$$I_m = \hat{\Gamma}_m A_k \bar{A}_p^2 \exp(i\sigma T_1) \quad (4.5b)$$

$$I_p = \hat{\Gamma}_p A_k \bar{A}_m \bar{A}_p \exp(i\sigma T_1) \quad (4.5c)$$

$$\hat{\Gamma}_k = \Gamma_{kmpp} + \Gamma_{kpmp} + \Gamma_{kppm} \quad (4.6a)$$

$$\hat{\Gamma}_m = \Gamma_{mppk} + \Gamma_{mpkp} + \Gamma_{mkpp} \quad (4.6b)$$

$$\hat{\Gamma}_p = \Gamma_{pmpk} + \Gamma_{ppmk} + \Gamma_{ppkm} + \Gamma_{pmkp} + \Gamma_{pkpm} + \Gamma_{pkmp} \quad (4.6c)$$

When the form of equation (2.8a) is

$$\omega_k = 3\omega_m + \epsilon\sigma \quad (4.7)$$

it follows that

$$I_k = \hat{\Gamma}_k A_m^3 \exp(-i\sigma T_1) \quad (4.8a)$$

$$I_m = \hat{\Gamma}_m A_k \bar{A}_m^2 \exp(i\sigma T_1) \quad (4.8b)$$

where

$$\hat{\Gamma}_k = \Gamma_{kmmm} \quad (4.9a)$$

$$\hat{\Gamma}_m = \Gamma_{mmmk} + \Gamma_{mmkm} + \Gamma_{mkmm} \quad (4.9b)$$

When the form of equation (2.8a) is

$$\omega_k = \omega_m + \omega_p - \omega_q + \epsilon\sigma \quad (4.10)$$

it follows that

$$I_k = \hat{\Gamma}_k A_m A_p \bar{A}_q \exp(-i\sigma T_1) \quad (4.11a)$$

$$I_m = \hat{\Gamma}_m A_k \bar{A}_p A_q \exp(i\sigma T_1) \quad (4.11b)$$

$$I_p = \hat{\Gamma}_p A_k \bar{A}_m A_q \exp(i\sigma T_1) \quad (4.11c)$$

$$I_q = \hat{\Gamma}_q A_m A_p \bar{A}_k \exp(-i\sigma T_1) \quad (4.11d)$$

where  $\hat{\Gamma}_k$ ,  $\hat{\Gamma}_m$ ,  $\hat{\Gamma}_p$ , and  $\hat{\Gamma}_q$  are given by equations (4.3).

Other possible internal resonances can be treated in a similar way.

#### 4.1 The Case Where $\lambda \approx 2\omega_m$ and $\omega_k \approx 3\omega_m$ .

The first case to be considered is  $\lambda \approx 2\omega_m$ ,  $\omega_k \approx 3\omega_m$ . The detuning parameters are defined in the following equations.

$$\omega_k = 3\omega_m + \epsilon\sigma \quad (4.12a)$$

$$\lambda = 2\omega_m + \epsilon\rho = \omega_k - \omega_m + \epsilon(\rho - \sigma) \quad (4.12b)$$

$$\begin{aligned} E_m = & -f_{mk} A_k \exp[-i(\rho - \sigma)T_1] \\ & -f_{mm} \bar{A}_m \exp(i\sigma T_1) \end{aligned} \quad (4.13a)$$

$$E_k = -f_{km} A_m \exp[i(\rho - \sigma)T_1] \quad (4.13b)$$

$$I_m = \hat{\Gamma}_m A_k \bar{A}_m^2 \exp(i\sigma T_1) \quad (4.14a)$$

$$I_k = \hat{\Gamma}_k A_m^3 \exp(-i\sigma T_1) \quad (4.14b)$$

Using the polar form (3.1), substituting (4.13) and (4.14) into equation (2.9), neglecting all except the m-th and the k-th modes, and separating the results into real and imaginary parts, we obtain

$$\begin{aligned} \omega_m(a_m' + c_m a_m) - \frac{\hat{\Gamma}_m}{8} a_k a_m^2 \sin \mu + \frac{f_{mm} a_m}{2} \sin \beta \\ - \frac{f_{mk}}{2} a_k \sin(\beta - \mu) = 0 \end{aligned} \quad (4.15a)$$

$$\begin{aligned} \omega_m a_m \alpha_m' + \frac{1}{8} a_m (\gamma_{mm} a_m^2 + \gamma_{mk} a_k^2) + \frac{\hat{\Gamma}_m}{8} a_k a_m^2 \cos \mu \\ - \frac{f_{mm} a_m}{2} \cos \beta - \frac{f_{mk}}{2} a_k \cos(\beta - \mu) = 0 \end{aligned} \quad (4.15b)$$

$$\omega_k(a_k' + c_k a_k) + \frac{\hat{\Gamma}_k}{8} a_m^3 \sin \mu + \frac{f_{km} a_m}{2} \sin(\beta - \mu) = 0 \quad (4.16a)$$

$$\begin{aligned} \omega_k a_k \alpha_k' + \frac{a_k}{8} (\gamma_{km} a_m^2 + \gamma_{kk} a_k^2) + \frac{\hat{\Gamma}_k}{8} a_m^3 \cos \mu \\ - \frac{f_{km} a_m}{2} \cos(\beta - \mu) = 0 \end{aligned} \quad (4.16b)$$

where

$$\mu = \sigma T_1 - 3\alpha_m + \alpha_k \quad (4.17a)$$

$$\beta = \rho T_1 - 2\alpha_m \quad (4.17b)$$

For the steady-state solution it follows that  $a_m = a_k = 0$  is a solution.

For nontrivial solutions

$$\mu' = \sigma + \alpha_k' - 3\alpha_m' = 0 \quad (4.18a)$$



$$\beta' = \rho - 2\alpha_m' = 0 \quad (4.18b)$$

which leads to

$$\alpha_m' = \frac{\rho}{2} \quad \text{and} \quad \alpha_k' = \frac{3\rho}{2} - \sigma \quad (4.19)$$

Equations (4.16) become

$$\omega_m C_m a_m + \frac{f_{mm} a_m}{2} \sin \beta - \frac{f_{mk}}{2} a_k \sin(\beta - \mu) - \frac{1}{8} \hat{\Gamma}_m a_k a_m^2 \sin \mu = 0 \quad (4.20a)$$

$$\begin{aligned} \omega_m \frac{\rho a_m}{2} + \frac{\gamma_{mm}}{8} a_m^3 + \frac{\gamma_{mk} a_m}{8} a_k^2 - \frac{f_{mm} a_m}{2} \cos \beta - \frac{f_{mk} a_k}{2} \cos(\beta - \mu) \\ + \frac{1}{8} \hat{\Gamma}_m a_k a_m^2 \cos \mu = 0 \end{aligned} \quad (4.20b)$$

$$\omega_k C_k a_k + \frac{f_{km} a_m}{2} \sin(\beta - \mu) + \frac{1}{8} \hat{\Gamma}_k a_m^3 \sin \mu = 0 \quad (4.20c)$$

$$\begin{aligned} \omega_k a_k \left( \frac{3\rho}{2} - \sigma \right) + \frac{\gamma_{mk}}{8} a_k a_m^2 + \frac{\gamma_{kk}}{8} a_k^3 - \frac{f_{km} a_m}{2} \cos(\beta - \mu) \\ + \frac{1}{8} \hat{\Gamma}_k a_m^3 \cos \mu = 0 \end{aligned} \quad (4.20d)$$

A detailed discussion of the behavior of the solution of equations (4.20) is given in Chapter 8 in the numerical examples.

#### 4.2 The Case Where $\lambda \approx 2\omega_k$ and $\omega_k \approx 3\omega_m$ .

The next case to be considered is  $\lambda \approx 2\omega_k$ ,  $\omega_k \approx 3\omega_m$ . The detuning parameters are defined as follows:

$$\lambda = 2\omega_k + \epsilon\rho \quad (4.22a)$$

$$\omega_k = 3\omega_m + \epsilon\sigma \quad (4.22b)$$

The parametric resonant terms are of the form

$$E_m = 0 \quad (4.22c)$$

$$E_k = -f_{kk} \bar{A}_k \exp(i\rho T_1) \quad (4.22d)$$

and the internal resonant terms are the same as those in equations (4.14). Substituting (4.22cd) and (4.14) into equation (2.9) and separating the result into real and imaginary parts, we find that

$$\omega_m(a'_m + C_m a_m) - \frac{1}{8} \hat{\Gamma}_m a_k a_m^2 \sin \mu = 0 \quad (4.23a)$$

$$\omega_m a_m \alpha'_m + \frac{a_m}{8} (\gamma_{mm} a_m^2 + \gamma_{mk} a_k^2) + \frac{1}{8} \hat{\Gamma}_m a_k a_m^2 \cos \mu = 0 \quad (4.23b)$$

$$\omega_k(a'_k + C_k a_k) + \frac{f_{kk} a_k}{2} \sin \beta + \frac{1}{8} \hat{\Gamma}_k a_m^3 \sin \mu = 0 \quad (4.23c)$$

$$\omega_k a_k \alpha'_k + \frac{a_k}{8} (\gamma_{km} a_m^2 + \gamma_{kk} a_k^2) - \frac{f_{kk} a_k}{2} \cos \beta + \frac{1}{8} \hat{\Gamma}_k a_m^3 \cos \mu = 0 \quad (4.23d)$$

where

$$\beta = \rho T_1 - 2\alpha_k \quad (4.24a)$$

$$\mu = \sigma T_1 + \alpha_k - 3\alpha_m \quad (4.24b)$$

For steady state solutions, it is easily seen that  $a_m = a_k = 0$  is a solution. For nontrivial solutions equations (4.24) become

$$\beta' = \rho - 2\alpha'_k = 0 \quad (4.25a)$$

$$\mu' = \sigma + \alpha'_k - 3\alpha'_m = 0 \quad (4.25b)$$

so that

$$\alpha'_k = \frac{\rho}{2} \quad \text{and} \quad \alpha'_m = \frac{\sigma + \frac{\rho}{2}}{3} \quad (4.26a)$$

and equations (4.23) become

$$\omega_m C_m - \frac{1}{8} \hat{\Gamma}_m a_k a_m \sin \mu = 0 \quad (4.27a)$$

$$\omega_m \frac{\sigma + \frac{\rho}{2}}{3} + \frac{\gamma_{mm} a_m^2}{8} + \frac{\gamma_{mk} a_k^2}{8} + \frac{1}{8} \hat{\Gamma}_m a_k a_m \cos \mu = 0 \quad (4.27b)$$

$$\omega_k C_k a_k + \frac{f_{kk} a_k}{2} \sin \beta + \frac{1}{8} \hat{\Gamma}_k a_m^3 \sin \mu = 0 \quad (4.27c)$$

$$\omega_k a_k \frac{\rho}{2} + \frac{\gamma_{km}}{8} a_k a_m^2 + \frac{\gamma_{kk}}{8} a_k^3 - \frac{f_{kk} a_k}{2} \cos \beta + \frac{1}{8} \hat{\Gamma}_k a_m^3 \cos \mu = 0 \quad (4.27d)$$

A detailed discussion of the behavior of the solutions of equations (4.27) is given in Chapter 8 in the numerical examples.

It should be noted here that it is possible for  $a_m = 0$  and  $a_k \neq 0$ , in which case equations (4.27c,d) become

$$\omega_k C_k a_k + \frac{f_{kk} a_k}{2} \sin \beta = 0 \quad (4.28a)$$

and

$$\omega_k a_k \frac{\rho}{2} + \frac{\gamma_{kk}}{8} a_k^3 - \frac{f_{kk} a_k}{2} \cos \beta = 0 \quad (4.28b)$$

This is exactly the same type of response that occurs when there is no internal resonance, equations (3.7a,b). Thus, the only mode which responds is the one which is directly excited. This is in contrast to the case where the frequency of the excitation  $\lambda$  is near  $2\omega_m$  and a single-mode response is not possible.



#### 4.3 The Case Where $\lambda \approx 4\omega_m$ and $\omega_k \approx 3\omega_m$ .

The last case to be considered is for  $\lambda \approx 4\omega_m$  and  $\omega_k \approx 3\omega_m$ . In the presence of this internal resonance the parametric resonance can be expressed equally well as  $\lambda \approx \omega_m + \omega_k$ . We may write

$$\lambda = 4\omega_m + \varepsilon\rho = \omega_m + \omega_k + \varepsilon(\rho - \sigma) \quad (4.29a)$$

where

$$\omega_k = 3\omega_m + \varepsilon\sigma \quad (4.29b)$$

The terms involved in the parametric resonance are

$$E_m = -f_{mk}\bar{A}_k \exp[i(\rho - \sigma)T_1] \quad (4.30a)$$

$$E_k = -f_{km}\bar{A}_m \exp[i(\rho - \sigma)T_1] \quad (4.30b)$$

Substituting equations (4.30) and (4.14) into equation (2.9), introducing the polar form (3.1), and separating the result into real and imaginary parts, we obtain

$$\omega_m(a'_m + C_m a_m) + \frac{f_{mk}a_k}{2} \sin\beta - \frac{1}{8} \hat{\Gamma}_m a_k a_m^2 \sin\mu = 0 \quad (4.31a)$$

$$\begin{aligned} \omega_m a_m \alpha'_m + \frac{1}{8} a_m (\gamma_{mm} a_m^2 + \gamma_{mk} a_k^2) - \frac{f_{mk}a_k}{2} \cos\beta \\ + \frac{1}{8} \hat{\Gamma}_m a_k a_m^2 \cos\mu = 0 \end{aligned} \quad (4.31b)$$

$$\omega_k(a'_k + C_k a_k) + \frac{f_{km}a_m}{2} \sin\beta + \frac{1}{8} \hat{\Gamma}_k a_m^3 \sin\mu = 0 \quad (4.31c)$$

$$\begin{aligned} \omega_k a_k \alpha'_k + \frac{a_k}{8} (\gamma_{km} a_m^2 + \gamma_{kk} a_k^2) - \frac{f_{km}a_m}{2} \cos\beta \\ + \frac{1}{8} \hat{\Gamma}_k a_m^3 \cos\mu = 0 \end{aligned} \quad (4.31d)$$

where

$$\beta = (\rho - \sigma) T_1 - \alpha_k - \alpha_m \quad (4.32a)$$

$$\mu = \sigma T_1 + \alpha_k - 3\alpha_m \quad (4.32b)$$

For the steady-state solution, it follows that  $a_m = a_k = 0$ . For the nontrivial solution,

$$\beta' = \rho - \sigma - \alpha_k' - \alpha_m' = 0 \quad (4.33a)$$

and

$$\mu' = \sigma + \alpha_k' - 3\alpha_m' = 0 \quad (4.33b)$$

which lead to

$$\alpha_m' = \frac{\rho}{4} \quad \text{and} \quad \alpha_k' = \frac{3\rho}{4} - \sigma \quad (4.33c)$$

The steady-state equations can be written as

$$\omega_m C_m a_m + \frac{f_{mk} a_k}{2} \sin \beta - \frac{1}{8} \hat{\Gamma}_m a_k a_m^2 \sin \mu = 0 \quad (4.34a)$$

$$\begin{aligned} \omega_m a_m \frac{\rho}{4} + \frac{a_m^3 \gamma_{mm}}{8} + \frac{a_m a_k^2 \gamma_{mk}}{8} - \frac{f_{mk} a_k}{2} \cos \beta \\ + \frac{1}{8} \hat{\Gamma}_m a_k a_m^2 \cos \mu = 0 \end{aligned} \quad (4.34b)$$

$$\omega_k C_k a_k + \frac{f_{km} a_m}{2} \sin \beta + \frac{1}{8} \hat{\Gamma}_k a_m^3 \sin \mu = 0 \quad (4.34c)$$

$$\begin{aligned} \omega_k a_k \frac{3\rho - 4\sigma}{4} + \frac{a_k^3 \gamma_{kk}}{8} + \frac{a_k a_m^2 \gamma_{km}}{8} - \frac{f_{km} a_m}{2} \cos \beta \\ + \frac{1}{8} \hat{\Gamma}_k a_m^3 \cos \mu = 0 \end{aligned} \quad (4.34d)$$

A detailed discussion of the response predicted by these equations is given in Chapter 8.

#### 4.4 Summary

Internal-resonance terms involving  $\omega_k \approx 3\omega_m$  have been developed. The response in the presence of this internal resonance and several parametric resonances has been examined. The following comments are pertinent:

a. It has been shown that, even in the presence of internal resonance, it is possible for only the mode which is excited to enter the response. The behavior is then essentially identical to that in the absence of internal resonance.

b. In all cases discussed, strong modal interactions due to the presence of internal resonance are possible. A full discussion of the dominance of one mode over the others is presented in Chapter 8, which contains the numerical examples.



## CHAPTER FIVE

### THE ANALYSIS OF SYSTEMS HAVING REPEATED FREQUENCIES IN THE ABSENCE OF INTERNAL AND PARAMETRIC RESONANCES

In the last three chapters we have discussed the response of a nonlinear system subject to a parametric excitation. Throughout the entire analysis we have considered the natural frequencies to be distinct. This is valid for the beam-column we discuss in Chapter 8 as well as for many other physical systems. The analysis contained in the next three chapters deals with the situation where one of the natural frequencies is repeated. As an example, this occurs in plates subject to aerodynamic loading at the onset of flutter. Two natural structural frequencies merge due to aeroelastic coupling. For a discussion of the mechanism of flutter the reader is referred to Dowell (1975).

In this chapter we follow a format similar to that of Chapter 2. The governing equations (2.1) are modified to account for the repeated frequency and the aerodynamic loading. In addition, scaling factors will be introduced in the form  $\epsilon^{\delta_i}$  to insure that terms interact at the proper order. The method of multiple scales is utilized to generate the solvability conditions. The conditions that predict the onset of flutter are developed and limit-cycle behavior is demonstrated. We begin now with the governing equations when one natural frequency,  $\omega_j$ , is repeated, thus we write

$$\begin{aligned}
\ddot{U}_n + \omega_n^2 U_n &= \epsilon^{\delta_F} [-(2 \cos \lambda t) \sum_{m=1}^{\infty} f_{nm} U_m] \\
&- \epsilon^{\delta_C} [2C_n \dot{U}_n] + \epsilon^{\delta_N} \left[ \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \Gamma_{nmpq} U_m U_p U_q \right] \\
&+ \epsilon^{\delta_\Lambda} \left[ \Lambda \sum_{m=1}^{\infty} \beta_{nm} U_m \right] \quad n = 1, 2, \dots \quad n \neq k \quad (5.1a)
\end{aligned}$$

and

$$\begin{aligned}
\ddot{U}_k + \omega_j^2 U_k + U_j &= \epsilon^{\delta_F} [-(2 \cos \lambda t) \sum_{m=1}^{\infty} f_{km} U_m] \\
&- \epsilon^{\delta_C} [2C_k \dot{U}_k] + \epsilon^{\delta_N} \left[ \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \Gamma_{kmpq} U_m U_p U_q \right] \\
&+ \epsilon^{\delta_\Lambda} \left[ \Lambda \sum_{m=1}^{\infty} \beta_{km} U_m \right] \quad \text{where } k = j + 1 \quad (5.1b)
\end{aligned}$$

where  $k = j + 1$ ,  $\Lambda$  is a detuning parameter (associated with the aerodynamic loading) to be explained later in a numerical example, and  $\delta_F$ ,  $\delta_C$ ,  $\delta_N$ , and  $\delta_\Lambda$  are constants to be chosen so as to insure the proper interaction for the various resonances to be considered.

We note that, if the nonlinear, damping and forcing terms were zero, the  $k$ -th mode would grow indefinitely, and the system would be in flutter. In the present situation, we expect the  $k$ -th mode to have a much larger amplitude than the  $j$ -th mode, and we have no indication of the other amplitudes. Thus, using the method of multiple scales, we obtain a uniformly valid approximation of the solution of equations (5.1) in the form

$$\begin{aligned}
U_n &= \epsilon^{-\delta_n} [U_{n_0}(T_0, T_1, T_2) + \epsilon^{\delta_0} U_{n_1}(T_0, T_1, T_2) \\
&+ \epsilon^{2\delta_0} U_{n_2}(T_0, T_1, T_2)] + \dots \quad (5.2a)
\end{aligned}$$

where the  $\delta_n$  and  $\delta_0$  are more constants to be determined, and

$$T_n = \epsilon^{n\delta_0} t \quad (5.2b)$$

The derivatives become

$$\frac{d}{dt} = D_0 + \epsilon^{\delta_0} D_1 + \epsilon^{2\delta_0} D_2 + \dots \quad (5.3a)$$

and

$$\frac{d^2}{dt^2} = D_0^2 + 2\epsilon^{\delta_0} D_0 D_1 + \epsilon^{2\delta_0} (D_1^2 + 2D_0 D_2) + \dots \quad (5.3b)$$

where

$$D_n \equiv \frac{\partial}{\partial T_n} \quad (5.3c)$$

Substituting equations (5.2) and (5.3) into equations (5.1) yields

$$\begin{aligned} & D_0^2 U_{n_0} + \omega_n^2 U_{n_0} + \epsilon^{\delta_0} [D_0^2 U_{n_1} + \omega_n^2 U_{n_1} + 2D_0 D_1 U_{n_0}] \\ & + \epsilon^{2\delta_0} [D_0^2 U_{n_2} + \omega_n^2 U_{n_2} + 2D_0 D_2 U_{n_0} + D_1^2 U_{n_0} + 2D_0 D_1 U_{n_1}] \\ & + \Delta_{kn} [\epsilon^{\delta_k - \delta_j} U_{j_0} + \epsilon^{\delta_0 + \delta_k - \delta_j} U_{j_1} + \epsilon^{2\delta_0 + \delta_k - \delta_j} U_{j_2}] \\ & + \epsilon^{\delta_C} [2C_n D_0 U_{n_0}] + \epsilon^{\delta_C + \delta_0} [2C_n (D_0 U_{n_1} + D_1 U_{n_0})] \\ & + \epsilon^{\delta_C + 2\delta_0} [2C_n (D_2 U_{n_0} + D_1 U_{n_1}) - [\Lambda \sum_{m=1}^{\infty} \beta_{nm} (U_{m_0} + \epsilon^{\delta_0} U_{m_1} \\ & + \epsilon^{2\delta_0} U_{m_2}) \epsilon^{\delta_\Lambda + \delta_n - \delta_m}] + [\sum_{m=1}^{\infty} f_{nm} (U_{m_0} + \epsilon^{\delta_0} U_{m_1} + \epsilon^{2\delta_0} U_{m_2}) \times \\ & \epsilon^{\delta_F + \delta_n - \delta_m}] (2\cos\lambda T_0) - \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \Gamma_{nmpq} (U_{m_0} + \epsilon^{\delta_0} U_{m_1} \\ & + \epsilon^{2\delta_0} U_{m_2}) \times (U_{p_0} + \epsilon^{\delta_0} U_{p_1} + \epsilon^{2\delta_0} U_{p_2}) \times (U_{q_0} + \epsilon^{\delta_0} U_{q_1} \\ & + \epsilon^{2\delta_0} U_{q_2}) \times \epsilon^{\delta_N + \delta_n - \delta_m - \delta_p - \delta_q} = 0, \quad n = 1, 2, \dots \quad (5.4a) \end{aligned}$$



where

$$k = j + 1 \quad (5.4b)$$

$$\omega_j = \omega_k \quad (5.4c)$$

and

$$\Delta_{kn} = \begin{cases} 1 & \text{when } n = k \\ 0 & \text{otherwise} \end{cases} \quad (5.4d)$$

Equation (5.4a) is the basis for the analysis of systems having a repeated frequency, and we refer to it again in subsequent chapters.

In the present chapter, we consider the case in which there is neither a parametric nor an internal resonance. Thus, here we consider the flutter condition for the linear system and obtain the limit-cycle behavior of the nonlinear system. In this chapter, we need to use only a two-term expansion, but in Chapter 6 where various parametric resonances are studied, we need a three-term expansion.

For a reference, we use the  $j$ -th mode and hence let

$$\delta_j = 0 \quad (5.5)$$

In order to have the effect of damping included in a first approximation, we must let

$$\delta_C = \delta_0 \quad (5.6a)$$

And to account for the repeated frequency here, we must let

$$\delta_k - \delta_j = \delta_0 \quad \text{or simply} \quad \delta_k = \delta_0 \quad (5.6b)$$

We presume that the  $j$ -th and  $k$ -th modes are the ones primarily involved in the representation of the flutter system. Thus we focus on these mode.

It appears that to bring the effect of the aerodynamic loading into the first approximation, we must put

$$\delta_{\Lambda} + \delta_n - \delta_m = \delta_0$$

We note that, when  $n=j$  and  $m=k$ , this statement leads to

$$\delta_{\Lambda} = 2\delta_0 \quad (5.7)$$

and when  $n=k$  and  $m=j$ , it leads to

$$\delta_{\Lambda} = 0$$

If  $\delta_{\Lambda} = 0$ , the aerodynamic term in the equation governing the  $U_j$  is elevated to  $O(\varepsilon^{-\delta_0})$ , which is unacceptable. On the other hand, if  $\delta_{\Lambda} = 2\delta_0$ , such unacceptable result follows; thus, we let  $\delta_{\Lambda} = 2\delta_0$ .

To account for the nonlinear terms in the first approximation, we must put

$$\delta_N + \delta_n - \delta_m - \delta_p - \delta_q = \delta_0$$

When  $n=j$  and  $m=p=q=k$ , this leads to

$$\delta_N = 4\delta_0 \quad (5.8)$$

For other combinations of modes, one obtains different results for  $\delta_N$ , but these all lead to inconsistencies.

Here we do not consider the effect of a parametric excitation. Thus, for the time being, we consider

$$f_{mn} = 0 \quad \text{for all } n \text{ and } m \quad (5.9)$$

Finally, when  $n \neq j$  or  $k$ , we let

$$0 \leq \delta_n \leq \delta_0 \quad (5.10)$$

### 5.1 The Case of Modes Having Repeated Frequencies

#### A. The Linear Response - The Flutter Condition

In the absence of both an internal and a parametric resonance the only modes remaining in the analysis are the  $j$ -th and  $k$ -th modes.

For the linear response, we let

$$\Gamma_{nmpq} = 0 \quad \text{for all } n, m, p, q \quad (5.11)$$

Substituting equations (5.5) thru (5.11) into equation (5.4) and equating coefficients of  $\epsilon^{n\delta_0}$ , we obtain

$$D_0^2 U_{j0} + \omega_j^2 U_{j0} = 0 \quad (5.12a)$$

$$D_0^2 U_{k0} + \omega_k^2 U_{k0} = 0 \quad (5.12b)$$

$$D_0^2 U_{j1} + \omega_j^2 U_{j1} = -2D_0 D_1 U_{j0} - 2C_j D_0 U_{j0} + \Lambda \beta_{jk} U_{k0} \quad (5.12c)$$

$$D_0^2 U_{k1} + \omega_k^2 U_{k1} = -2D_0 D_1 U_{k0} - U_{j0} - 2C_k D_0 U_{k0} \quad (5.12d)$$

The solution of equations (5.12a) and (5.12b) can be written in the form

$$U_{j0} = A_j(T_1) \exp(i\omega_j T_0) + cc \quad (5.13a)$$

$$U_{k0} = A_k(T_1) \exp(i\omega_k T_0) + cc \quad (5.13b)$$

Substituting equations (5.13) into equations (5.12c) and (5.12d), and separating the result into real and imaginary parts, we obtain the following conditions for the elimination of secular terms



$$- 2i\omega_j(A_j' + C_j A_j) + \Lambda \beta_{jk} A_k = 0 \quad (5.14a)$$

$$- 2i\omega_j(A_k' + C_k A_k) - A_j = 0 \quad (5.14b)$$

Writing  $A_j$  and  $A_k$  as

$$A_j = B_j \exp(\lambda T_1) \quad \text{and} \quad A_k = B_k \exp(\lambda T_1) \quad (5.15)$$

and substituting into equation (5.15) into equations (5.14) leads to

$$2i\omega_j(\lambda + C_j)B_j - \Lambda \beta_{jk} B_k = 0 \quad (5.16a)$$

$$B_j' + 2i\omega_j(\lambda + C_k)B_k = 0 \quad (5.16b)$$

Setting the determinant of the coefficients equal to zero, we obtain

$$\begin{vmatrix} 2i\omega_j(\lambda + C_j) & -\Lambda \beta_{jk} \\ 1 & 2i\omega_j(\lambda + C_k) \end{vmatrix} = 0 \quad (5.17a)$$

or

$$\lambda^2 + (C_j + C_k)\lambda + (C_j C_k - \frac{\Lambda \beta_{jk}}{4\omega_j^2}) = 0 \quad (5.17b)$$

Hence

$$\lambda = -\frac{C_j + C_k}{2} \pm \frac{1}{2} \sqrt{(C_j - C_k)^2 + \frac{\Lambda \beta_{jk}}{\omega_j^2}} \quad (5.17c)$$

From (5.17c) we see that the solution decays if

$$(C_j + C_k)^2 > (C_j - C_k)^2 + \frac{\Lambda \beta_{jk}}{\omega_j^2} \quad (5.18a)$$

or

$$\Lambda > 4\omega_j^2 C_j C_k / \beta_{jk} \quad (5.18b)$$

is the condition for flutter. Thus in the absence of damping, any value of  $\Lambda$  will cause flutter. Here we should note that  $\beta_{jk}$  is a positive aerodynamic coefficient that is associated with the so-called "piston-theory" approximation. Then  $\Lambda$  is a detuning parameter that is associated with the flutter speed;  $\Lambda$  is zero when the airspeed equals the flutter speed for the undamped system. When  $\Lambda$  is greater than zero, the airspeed is above the flutter speed. A more detailed explanation will be given in the numerical example of Chapter 9.

#### B. The Nonlinear Response - Limit Cycle Behavior

To examine the effect of the nonlinearity on the response, we no longer use equation (5.11). Again substituting equations (5.5) thru (5.10) into equation (5.4a) and equating coefficients of  $\epsilon^{n\delta_0}$ , we obtain

$$D_0^2 U_{j0} + \omega_j^2 U_{j0} = 0 \quad (5.19a)$$

$$D_0 U_{k0} + \omega_k^2 U_{k0} = 0 \quad (5.19b)$$

$$D_0^2 U_{j1} + \omega_j^2 U_{j1} = -2D_0 D_1 U_{j0} - 2C_j D_0 U_{j0} + \Gamma_{jkkk} U_{k0}^3 + \Lambda \beta_{jk} U_{k0} \quad (5.19c)$$

$$D_0^2 U_{k1} + \omega_k^2 U_{k1} = -2D_0 D_1 U_{k0} - 2C_k D_0 U_{k0} + U_{j0} \quad (5.19d)$$

The solution of equations (5.19a) and (5.19b) is equations (5.13). Substituting equations (5.13) into equations (5.19c) and (5.19d), introducing the polar form (3.1), and separating the result into real

and imaginary parts, we obtain the following conditions for the elimination of secular terms:

$$\omega_j(a'_j + C_j a_j) + \left(\frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda \beta_{jk}}{2} a_k\right) \sin \gamma = 0 \quad (5.20a)$$

$$\omega_j(a'_k + C_k a_k) + \frac{1}{2} a_j \sin \gamma = 0 \quad (5.20b)$$

$$\alpha'_j a_j \omega_j + \left(\frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda \beta_{jk}}{2} a_k\right) \cos \gamma = 0 \quad (5.20c)$$

$$\alpha'_k a_k \omega_j - \frac{1}{2} a_j \cos \gamma = 0 \quad (5.20d)$$

where

$$\gamma = \alpha_j - \alpha_k \quad (5.21a)$$

For steady-state solutions  $a'_j = a'_k = 0$ . We see immediately that  $a_j = a_k = 0$  are possible solutions. For nontrivial solutions,  $a_j \neq a_k \neq 0$ , it follows that  $\gamma' = 0$  then

$$\alpha'_j = \alpha'_k \quad (5.21b)$$

The equations governing the steady-state response then become

$$\omega_j C_j a_j + \left(\frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda \beta_{jk}}{2} a_k\right) \sin \gamma = 0 \quad (5.22a)$$

$$\alpha'_j a_j \omega_j + \left(\frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda \beta_{jk}}{2} a_k\right) \cos \gamma = 0 \quad (5.22b)$$

$$\omega_j C_k a_k + \frac{a_j \sin \gamma}{2} = 0 \quad (5.22c)$$

$$\alpha'_k a_k \omega_j - \frac{a_j \cos \gamma}{2} = 0 \quad (5.22d)$$

We now multiply (5.22a) by  $C_k a_k$  and (5.22c) by  $C_j a_j$ , subtract, and obtain



$$\left[ \left( \frac{3}{8} \Gamma_{jkkk} a_k^2 + \frac{\Lambda \beta_{jk}}{2} \right) C_k a_k^2 - \frac{C_j a_j^2}{2} \right] \sin \gamma = 0 \quad (5.23a)$$

Similarly we multiply (5.22b) by  $a_k$  and (5.22d) by  $a_j$ , subtract, and obtain

$$\left[ \left( \frac{3}{8} \Gamma_{jkkk} a_k^2 + \frac{\Lambda \beta_{jk}}{2} \right) a_k^2 + \frac{a_j^2}{2} \right] \cos \gamma = 0 \quad (5.23b)$$

The coefficient of  $\sin \gamma$  in equation (5.23a) and the coefficient of  $\cos \gamma$  in equation (5.23b) cannot be zero simultaneously. Moreover,  $\sin \gamma$  and  $\cos \gamma$  cannot be zero simultaneously. It follows from equation (5.22c) that

$$\sin \gamma \neq 0 \quad (5.24a)$$

Thus, we must have

$$\cos \gamma = 0 \quad (5.24b)$$

and

$$\left( \frac{3}{4} \Gamma_{jkkk} a_k^2 + \Lambda \beta_{jk} \right) C_k a_k^2 - C_j a_j^2 = 0 \quad (5.24c)$$

If  $\cos \gamma = 0$ , then from (5.22c) it follows that

$$\sin \gamma = -1 \quad (5.24d)$$

Substituting (5.24d) into (5.22c) leads to

$$a_j = 2\omega_j C_k a_k \quad (5.25a)$$

Then substituting (5.25a) into (5.24c) leads to

$$a_k = 2 \sqrt{\frac{4\omega_j^2 C_j C_k - \Lambda \beta_{jk}}{3\Gamma_{jkkk}}} \quad (5.25b)$$

If  $\Gamma_{jkk}$  is negative (this is the case in the numerical example considered in Chapter 9), the condition for the existence of a real solution is

$$\Lambda > 4\omega_j^2 C_j C_k / \beta_{jk} \quad (5.25c)$$

in full agreement with equation (5.18b), which predicts the onset of flutter. The linear analysis merely predicts growth, while the nonlinear analysis predicts the development of a finite-amplitude limit cycle. The existence of limit cycles was found numerically, but not analytically, by Dowell (1966,1970). Smith and Morino (1976) developed limit cycles analytically but not in the presence of parametric excitation.

## 5.2 Summary

The analysis of linear and nonlinear systems having one repeated frequency has been carried out in the absence of either an internal or a parametric resonance. The amplitudes of all the modes not associated with the repeated frequency are shown to decay. For the two modes associated with the repeated frequency, the linear analysis predicts an infinitely growing solution if the condition

$$\Lambda > 4\omega_j^2 C_j C_k / \beta_{jk} \quad (5.26)$$

is met. The nonlinear analysis requires that the same condition be met if nontrivial solutions are to exist. However, the nonlinearity puts a bound on the response and limit-cycle behavior is found.

Now we can write a complete generalization of the solvability condition of equation (2.9) that is valid for the cases where either distinct or repeated frequencies exist. It has the form

$$- 2i\omega_n(D_1 A_n + C_n A_n) + A_n \sum_{p=1}^{\infty} \gamma_{np} A_p \bar{A}_p + I_n + E_n + K_n = 0 \quad (5.27a)$$

where

$$K_n = \begin{bmatrix} \sum_{p=1}^{\infty} \Lambda \beta_{np} A_p & \text{if } n \neq j, m \neq k \\ \sum_{p=1}^{\infty} \Lambda \beta_{jp} A_p + A_k \sum_{p=1}^{\infty} \gamma_{jp} A_p \bar{A}_p & \text{if } n = j \\ \sum_{p=1}^{\infty} \Lambda \beta_{kp} A_p + A_j \sum_{p=1}^{\infty} \gamma_{kp} A_p \bar{A}_p - A_j & \text{if } n = k \end{bmatrix} \quad (5.27b)$$

and

$$\gamma_{jp} = \begin{bmatrix} 3\Gamma_{jkkk} & \text{if } p = k \\ 2(\Gamma_{jkpp} + \Gamma_{jpkp} + \Gamma_{jp pk}) & \text{if } p \neq k \end{bmatrix} \quad (5.27c)$$

and

$$\gamma_{kp} = \begin{bmatrix} 3\Gamma_{kjjj} & \text{if } p = j \\ 2(\Gamma_{kjpp} + \Gamma_{kpjp} + \Gamma_{kppj}) & \text{if } p \neq j \end{bmatrix} \quad (5.27d)$$

It should be noted that for the case of distinct frequencies the  $K_n$  term is set equal to zero and equation (5.31a) becomes identically equation (2.9). When repeated frequencies are considered and appropriate scaling parameters,  $\delta_i$ , are specified by equations (5.5) through (5.11), then we note that in equation (5.27a) the term

$$A_n \sum_{p=1}^{\infty} \gamma_{np} A_p \bar{A}_p$$

is a higher-order term and hence does not enter the analysis at this level of approximation. In addition equation (5.27b) simplifies to



$$K_n = \begin{bmatrix} 0 & \text{if } n \neq j \text{ or } k \\ \Lambda \beta_{jk} A_k + 3\Gamma_{jkkk} A_k^2 \bar{A}_k & \text{if } n = j \\ -A_j & \text{if } n = k \end{bmatrix} \quad (5.27e)$$

## CHAPTER SIX

### PARAMETRIC RESONANCE IN THE ABSENCE OF INTERNAL RESONANCE

In this chapter we investigate several cases of parametric resonance in the absence of an internal resonance. We demonstrate that using a two-term expansion used in the method of multiple scales is inadequate and a three-term expansion is required for the combination resonance.

We recall from Chapter 5 that the modes not involved in an internal or a parametric resonance decay as  $t \rightarrow \infty$ . Here the emphasis is on steady-state solution; thus, we do not consider these modes further.

#### 6.1 The Case Where $\lambda \approx 2\omega_j$ .

The resonant condition associated with this case is

$$\lambda = 2\omega_j + \epsilon^\delta \rho \quad (6.1)$$

where  $\delta$  is a constant to be determined and  $\rho$  is a detuning parameter. Based on equations (5.27a) and (5.27e), we see that the solvability conditions for the elimination of secular terms have the form

$$-2i\omega_j(A_j' + C_j A_j) + I_j + E_j + 3\Gamma_{jkkk} A_k^2 \bar{A}_k + \Lambda_{jk} A_k \quad (6.2a)$$

$$-2i\omega_j(A_k' + C_k A_k) + I_k + E_k - A_j = 0 \quad (6.2b)$$

where  $I_n$  and  $E_n$  are terms resulting from internal and parametric resonances, respectively. In the last chapter, we analyzed these equations with  $I_n = E_n = 0$ . To examine the effect of the above resonant combination, we

must include the  $f_{jk}U_{k0}$  term from equation (5.4a) with  $n = j$ . We keep  $E_k = 0$  and  $I_j = I_k = 0$ . The condition for inclusion of this parametric resonant term is

$$\delta_F + \delta_j - \delta_k = \delta_0 \quad (6.3a)$$

Substituting equations (5.5) and (5.6b) into equation (6.3a) we have

$$\delta_F = 2\delta_0 \quad (6.3b)$$

Hence the parametric resonant term becomes

$$E_j = -f_{jk}\bar{A}_k \exp[(i\epsilon^{\delta_0\rho})T_0] \quad (6.4a)$$

Because the interaction occurs in the second terms of the expansion, we put

$$\delta = \delta_0 \quad (6.4b)$$

so that using equation (5.2b) we obtain

$$E_j = -f_{jk}\bar{A}_k \exp(i\rho T_1) \quad (6.4c)$$

The solvability conditions for the elimination of secular terms become

$$-i2\omega_j(A'_j + C_j A_j) - f_{jk}\bar{A}_k \exp(i\rho T_1) + 3\Gamma_{jkkk}A_k^2\bar{A}_k + \Lambda\beta_{jk}A_k = 0 \quad (6.5a)$$

$$-i2\omega_j(A'_k + C_k A_k) - A_j = 0 \quad (6.5b)$$

Introducing the polar form (3.1) and separating the result into real and imaginary parts, we obtain

$$\omega_j(a'_j + C_j a_j) + \frac{f_{jk}}{2} a_k \sin\beta + \left(\frac{3}{8} \Gamma_{jkkk} a_k^3 + \Lambda \frac{\beta_{jk} a_k}{2}\right) \sin\gamma = 0 \quad (6.6a)$$



$$\omega_j a_j \alpha'_j - \frac{f_{jk}}{2} a_k \cos \beta + \left( \frac{3}{8} \Gamma_{jkkk} a_k^3 + \Lambda \frac{\beta_{jk} a_k}{2} \right) \cos \gamma = 0 \quad (6.6b)$$

$$\omega_j (a'_k + C_k a_k) + \frac{a_j}{2} \sin \gamma = 0 \quad (6.6c)$$

$$\omega_j a_k \alpha'_k - \frac{a_j}{2} \cos \gamma = 0 \quad (6.6d)$$

where

$$\beta = \rho T_1 - \alpha_k - \alpha_j \quad (6.7a)$$

$$\gamma = \alpha_j - \alpha_k \quad (6.7b)$$

For steady-state conditions,  $a'_j = a'_k = 0$ . Either both  $a_j$  and  $a_k$  are zero, or neither is zero. For nontrivial solutions,

$$\beta' = \rho - \alpha'_j - \alpha'_k = 0 \quad (6.8a)$$

and

$$\gamma' = \alpha'_j - \alpha'_k = 0 \quad (6.8b)$$

which yield

$$\alpha'_j = \alpha'_k = \frac{\rho}{2} \quad (6.9)$$

The steady-state equations become

$$\omega_j C_j a_j + \frac{f_{jk}}{2} a_k \sin \beta + \left( \frac{3}{8} \Gamma_{jkkk} a_k^3 + \Lambda \frac{\beta_{jk} a_k}{2} \right) \sin \gamma = 0 \quad (6.10a)$$

$$\omega_j C_k a_k + \frac{a_j}{2} \sin \gamma = 0 \quad (6.10b)$$

$$\omega_j a_j \frac{\rho}{2} - \frac{f_{jk}}{2} a_k \cos \beta + \left( \frac{3}{8} \Gamma_{jkkk} a_k^3 + \Lambda \frac{\beta_{jk} a_k}{2} \right) \cos \gamma = 0 \quad (6.10c)$$

$$\omega_j a_k \rho - a_j \cos \gamma = 0 \quad (6.10d)$$

It is possible to combine equations (6.10b) and (6.10d) to obtain

$$a_j^2 = \omega_j^2 a_k^2 (\rho^2 + 4C_k^2) \quad (6.11a)$$

Equations (6.10b) and (6.10d) may be substituted into (6.10a) and (6.10c), respectively, so that, upon solving for  $\sin\beta$  and  $\cos\beta$  and squaring the results, we can use (6.11a) to obtain a closed-form solution:

$$a_k^2 = \frac{4}{3\Gamma_{jkkk}} \left[ -\Lambda\beta_{jk} + \omega_j^2 (4C_j C_k - \rho^2) \pm \sqrt{f_{jk}^2 - 4\omega_j^4 \rho^2 (C_j + C_k)^2} \right] \quad (6.11b)$$

We note that, if a nontrivial solution exists as  $f_{jk} \rightarrow 0$ , then  $\rho \rightarrow 0$  and hence equation (6.11b) reduces to

$$a_k = 2 \sqrt{\frac{4\omega_j^2 C_j C_k - \Lambda\beta_{jk}}{3\Gamma_{jkkk}}} \quad (6.11c)$$

and (6.11a) reduces to

$$a_j = 2\omega_j C_k a_k \quad (6.11d)$$

in full agreement with equation (5.25), which gives the amplitudes of the limit cycle for the unforced response.

More discussion of the results can be found in Chapter 9 where a numerical example is presented.

## 6.2 The Case Where $\lambda \approx \omega_r + \omega_j$ .

The resonant combination associated with this case is

$$\lambda = \omega_r + \omega_j + \epsilon^\delta \rho \quad (6.12)$$

To determine the effect of the resonant combination, we must include the following terms in (5.4a):  $f_{jr}U_{r0}$  when  $n = j$  and  $f_{rk}U_{k0}$  when  $n = r$ . We now have three equations to analyze since the  $r^{\text{th}}$ -mode has been drawn into the response by the resonant combination of equation (6.12). If the scaling conditions specified in equations (5.5) thru (5.10) are met along with equation (6.3b), we see that the resonant term of the form  $f_{jn}U_{r0}$  in equation (5.4a), for  $n = j$ , will be of higher order and will not enter the two-term expansion. Satisfying equation (5.10) for  $\delta_r = \delta_0$  allows for inclusion of the  $f_{jr}U_{r0}$  term; however the term  $f_{rk}U_{r0}$  in equation (5.4a), for  $n = r$ , will be of higher order and will not enter the two-term expansion. In this case the solution predicts that  $A_r$  decays as  $t \rightarrow \infty$ . Thus it appears that the original two-term expansion is an inadequate approximation, and recourse is made to a three-term expansion.

We re-examine equation (5.4a) and proceed with an analysis which parallels that of Chapter 5.

We arbitrarily choose the scaling coefficients,  $\epsilon^{\delta_0}$ , so that the effects of damping, the repeated frequency, the aerodynamic loading, the nonlinearity, and the parametric resonance all interact at the same order of approximation. We have already determined that this will not occur at  $O(\epsilon^{\delta_0})$ , so we expect that a uniformly valid solution will exist at  $O(\epsilon^{2\delta_0})$ . Thus the  $\delta$  in equation (6.12) is set at

$$\delta = 2\delta_0 \quad (6.13a)$$



For a reference, we again use the  $j$ -th mode and hence let

$$\delta_j = 0 \quad (6.13b)$$

In order to include the effect of damping, now we must let

$$\delta_c = 2\delta_0 \quad (6.13c)$$

Accounting for the effect of the repeated frequency, we must let

$$\delta_k - \delta_j = 2\delta_0$$

Using (6.13a) we obtain

$$\delta_k = 2\delta_0 \quad (6.13d)$$

To bring the aerodynamic loading into the approximation, we set

$$\delta_\Lambda + \delta_n - \delta_m = 2\delta_0,$$

where we note that for  $n = j$  and  $m = k$

$$\delta_\Lambda = 4\delta_0 \quad (6.13e)$$

and for  $n = k$  and  $m = j$

$$\delta_\Lambda = 0$$

Following the reasoning of Chapter 5, we conclude that equation (6.13e) must be satisfied to produce acceptable results.

To account for the nonlinearity, we set

$$\delta_N + \delta_n - \delta_m - \delta_p - \delta_q = 2\delta_0$$

When  $n = j$  and  $m = p = q = k$ , this leads to

$$\delta_N = 8\delta_0 \quad (6.13f)$$

Other combinations of  $n$ ,  $m$ ,  $p$ , and  $q$  lead to different values for  $\delta_N$ , but all these cases lead to inconsistencies in the expansion.

Finally, to include the effect of the parametric resonance, we set

$$\delta_F + \delta_n - \delta_m = 2\delta_0$$

Setting

$$\delta_F = 3\delta_0 \quad (6.13g)$$

allows for acceptable solutions. If  $n = j$  and  $m = r$ , then equation (6.13g) yields

$$\delta_r = \delta_0 \quad (6.13h)$$

and one of the parametric resonance terms  $f_{jr}U_{r0}$  can be accommodated. When  $n = r$  and  $m = k$  the other resonant terms  $f_{rk}U_{k0}$  is also included. Substituting equations (6.13) into equation (5.4a) and equating coefficients of equal powers of  $\epsilon^{n\delta_0}$ , we obtain

$$D_0^2 U_{j0} + \omega_j^2 U_{j0} = 0 \quad (6.14a)$$

$$D_0^2 U_{k0} + \omega_k^2 U_{k0} = 0 \quad (6.14b)$$

$$D_0^2 U_{r0} + \omega_r^2 U_{r0} = 0 \quad (6.14c)$$

$$D_0^2 U_{j1} + \omega_j^2 U_{j1} + 2D_0 D_1 U_{j0} + 2f_{jk} U_{k0} \cos \lambda T_0 = 0 \quad (6.15a)$$

$$D_0^2 U_{k1} + \omega_k^2 U_{k1} + 2D_0 D_1 U_{k0} = 0 \quad (6.15b)$$

$$D_0^2 U_{r1} + \omega_r^2 U_{r1} + 2D_0 D_1 U_{r0} = 0 \quad (6.15c)$$

$$\begin{aligned}
& D_0^2 U_{j2} + \omega_j^2 U_{j2} + 2D_0 D_2 U_{j0} + D_1^2 U_{j0} + 2D_0 D_1 U_{j1} + 2C_j D_0 U_{j0} \\
& + 2f_{jr} U_{r0} \cos \lambda T_0 + 2f_{jk} U_{k1} \cos \lambda T_0 - \Gamma_{jkkk} U_{k0}^3 \\
& - \Lambda \beta_{jk} U_{k0} = 0
\end{aligned} \tag{6.16a}$$

$$\begin{aligned}
& D_0^2 U_{k2} + \omega_k^2 U_{k2} + 2D_0 D_2 U_{k0} + D_1^2 U_{k0} + 2D_0 D_1 U_{k1} + 2C_k D_0 U_{k0} \\
& + U_{j0} = 0
\end{aligned} \tag{6.16b}$$

$$\begin{aligned}
& D_0^2 U_{r2} + \omega_r^2 U_{r2} + 2D_0 D_2 U_{r0} + D_1^2 U_{r0} + 2D_0 D_1 U_{r1} + 2C_r D_0 U_{r0} \\
& + 2f_{rk} U_{k0} \cos \lambda T_0 = 0
\end{aligned} \tag{6.16c}$$

The solutions of equations (6.14) are

$$U_{j0} = A_j(T_1, T_2) \exp(i\omega_j T_0) + cc \tag{6.17a}$$

$$U_{k0} = A_k(T_1, T_2) \exp(i\omega_k T_0) + cc \tag{6.17b}$$

$$U_{r0} = A_r(T_1, T_2) \exp(i\omega_r T_0) + cc \tag{6.17c}$$

Substituting equations (6.17) into equations (6.15), we find that secular terms are eliminated provided

$$D_1 A_j = D_1 A_k = D_1 A_r = 0 \tag{6.18}$$

and that the  $A_n$  are independent of  $T_1$ .

It follows that

$$U_{j1} = f_{jk} \left[ \frac{A_k \exp[i(\lambda + \omega_j)T_0]}{\lambda(\lambda + 2\omega_j)} + \frac{\bar{A}_k \exp[i(\lambda - \omega_j)T_0]}{\lambda(\lambda - 2\omega_j)} \right] + cc \tag{6.19a}$$



and

$$U_{k1} = U_{r1} = 0 \quad (6.19b)$$

Substituting (6.17) through (6.19) into (6.16) yields the solvability conditions

$$- 2i\omega_j(A'_j + C_j A_j) - f_{jr} \bar{A}_r \exp(i\rho T_2) + 3\Gamma_{jkkk} A_k^2 \bar{A}_k + \Lambda \beta_{jk} A_k = 0 \quad (6.20a)$$

$$- 2i\omega_j(A'_k + C_k A_k) - A_j = 0 \quad (6.20b)$$

$$- 2i\omega_r(A'_r + C_r A_r) - f_{rk} \bar{A}_k \exp(i\rho T_2) = 0 \quad (6.20c)$$

where primes denote differentiation with respect to  $T_2$ . Letting

$$A_j = \frac{1}{2} a_n(T_2) \exp[i\alpha_n(T_2)] \quad n = j, k, r \quad (6.21)$$

substituting equation (6.21) into equations (6.20) and separating the result into real and imaginary parts, we obtain

$$\omega_j(a'_j + C_j a_j) + \frac{f_{jr}}{2} a_r \sin\beta_1 + \left(\frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda \beta_{jk}}{2} a_k\right) \sin\gamma = 0 \quad (6.22a)$$

$$\omega_j a_j \alpha'_j - \frac{f_{jr}}{2} a_r \cos\beta_1 + \left(\frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda \beta_{jk}}{2} a_k\right) \cos\gamma = 0 \quad (6.22b)$$

$$\omega_j(a'_k + C_k a_k) + \frac{a_j}{2} \sin\gamma = 0 \quad (6.22c)$$

$$\omega_j a_k \alpha'_k - \frac{a_j}{2} \cos\gamma = 0 \quad (6.22d)$$

$$\omega_r(a'_r + C_r a_r) + \frac{f_{rk}}{2} a_k \sin\beta_2 = 0 \quad (6.22e)$$

$$\omega_r a_r \alpha'_r - \frac{f_{rk}}{2} a_k \cos\beta_2 = 0 \quad (6.22f)$$

where

$$\beta_1 = \rho T_2 - \alpha_r - \alpha_j \quad (6.23a)$$

$$\gamma = \alpha_j - \alpha_k \quad (6.23b)$$

$$\beta_2 = \rho T_2 - \alpha_r - \alpha_k \quad (6.23c)$$

For the steady-state response,  $a_j' = a_k' = a_r' = 0$  and  $\beta_1' = \gamma' = \beta_2' = 0$ .

Thus

$$\alpha_j' = \alpha_k' \quad (6.24a)$$

$$\alpha_r' + \alpha_j' = \alpha_r' + \alpha_k' = \rho \quad (6.24b)$$

$$\omega_j C_j a_j + \frac{f_{jr}}{2} a_r \sin \beta_1 + \left[ \frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda \beta_{jk}}{2} a_k \right] \sin \gamma = 0 \quad (6.25a)$$

$$\omega_j C_k a_k + \frac{a_j}{2} \sin \gamma = 0 \quad (6.25b)$$

$$\omega_r C_r a_r + \frac{f_{rk}}{2} a_k \sin \beta_2 = 0 \quad (6.25c)$$

$$\rho = \frac{a_j}{2\omega_j a_k} \cos \gamma + \frac{f_{rk} a_k}{2\omega_r a_r} \cos \beta_2 \quad (6.25d)$$

$$\rho = \frac{1}{\omega_j a_j} \left\{ \frac{f_{jr}}{2} a_r \cos \beta_1 - \left[ \frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda \beta_{jk}}{2} a_k \right] \cos \gamma \right\} + \frac{f_{rk} a_k}{2\omega_r a_r} \cos \beta_2 \quad (6.25e)$$

$$a_j^2 \cos \gamma = 2a_k \left\{ \frac{f_{jr}}{2} a_r \cos \beta_1 - \left[ \frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda \beta_{jk}}{2} a_k \right] \cos \gamma \right\} \quad (6.25f)$$

From (6.22) we see that in the steady-state if any one of the  $a_n = 0$  the other two are also zero.

These equations do not readily admit a closed-form solution. They are discussed further in Chapter 9, where a numerical example is presented. We note that only five equations are independent, equation (6.25f) being a combination of equations (6.25e) and (6.25d). A sixth equation can be obtained from equations (6.23):

$$\beta_2 = \beta_1 + \gamma \quad (6.25g)$$

### 6.3 The Case Where $\lambda \approx \omega_r - \omega_j$ .

In this case the ordering scheme turns out to be the same as that of the previous section. Hence, we write the resonant combination as

$$\lambda = \omega_r - \omega_j + \epsilon^{2\delta_0} \rho \quad (6.26)$$

The resonant terms become

$$E_j \exp(i\omega_j T_0) = f_{jr} A_r \exp[i(\omega_j - \epsilon^{2\delta_0} \rho) T_0] \quad (6.27a)$$

and

$$E_r \exp(i\omega_r T_0) = f_{rk} A_k \exp[i(\omega_r + \epsilon^{2\delta_0} \rho) T_0] \quad (6.27b)$$

with

$$\beta_1 = -\rho T_2 + \alpha_r - \alpha_j \quad (6.28a)$$

$$\gamma = \alpha_j - \alpha_k \quad (6.28b)$$

$$\beta_2 = \rho T_2 - \alpha_r + \alpha_k \quad (6.28c)$$

The governing equations have the same form as those of the last section. Hence the steady-state equations become



$$\omega_j C_j a_j + \frac{f_{jr}}{2} a_r \sin \beta_1 + \left[ \frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda \beta_{jk}}{2} a_k \right] \sin \gamma = 0 \quad (6.29a)$$

$$\omega_j C_j a_j + \frac{a_j}{2} \sin \gamma = 0 \quad (6.29b)$$

$$\omega_r C_r a_r + \frac{f_{rk}}{2} a_k \sin \beta_2 = 0 \quad (6.29c)$$

$$\rho = \frac{f_{rk} a_k}{2 \omega_r a_r} \cos \beta_2 + \frac{a_j}{2 \omega_j a_k} \cos \gamma \quad (6.29d)$$

$$\rho = \frac{f_{rk} a_k}{2 \omega_r a_r} \cos \beta_2 + \frac{1}{\omega_j a_j} \left\{ \frac{f_{jr}}{2} a_r \cos \beta_1 - \left[ \frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda \beta_{jk}}{2} a_k \right] \cos \gamma \right\} \quad (6.29e)$$

$$\frac{a_j \cos \gamma}{2 \omega_j a_k} = \frac{1}{\omega_j a_j} \left\{ \frac{f_{jr}}{2} a_r \cos \beta_1 - \left[ \frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda \beta_{jk}}{2} a_k \right] \cos \gamma \right\} \quad (6.29f)$$

#### 6.4 Summary

Several cases in which parametric resonances are involved have been investigated. The resonance case  $\lambda \approx 2\omega_j$  appears at first order, while the resonance cases  $\lambda \approx \omega_r \pm \omega_j$  appear at second order. Nothing conclusive is self-evident from a cursory examination of the steady-state equations. Analysis and discussion await the coverage of numerical examples in Chapter 9.

## CHAPTER SEVEN

### THE EFFECT OF INTERNAL RESONANCE

Again we choose to investigate an internal resonance of the form  $\omega_r \approx 3\omega_j$ . In this chapter we examine parametric resonances in the presence of this internal resonance. We also discuss the case where the internal resonance is present but there is no parametric resonance, the only excitation coming from the aerodynamic loading.

#### 7.1 The Case Where $\lambda \approx 2\omega_j$ and $\omega_r \approx 3\omega_j$ .

The resonances associated with this case are

$$\lambda = 2\omega_j + \epsilon^\delta \rho \quad (7.1a)$$

and

$$\omega_r = 3\omega_j + \epsilon^\delta \sigma \quad (7.1b)$$

In this case, the general form of the solvability conditions is as follows:

$$- 2i\omega_j(A'_j + C_j A_j) + I_j + E_j + 3\Gamma_{jkkk} A_k \bar{A}_k + \Lambda \beta_{jk} A_k = 0 \quad (7.2a)$$

$$- 2i\omega_j(A'_k + C_k A_k) + I_k + E_k - A_j = 0 \quad (7.2b)$$

$$- 2i\omega_r(A'_r + C_r A_r) + I_r + E_r = 0 \quad (7.2c)$$

To obtain the best approximation, we need to develop an ordering scheme which includes the effect of the repeated frequency, the parametric resonance, the internal resonance, and as many other terms as it is consistent to retain.

Proceeding as in Section 6.1, we find that equations (5.5) thru (5.10) and (6.3b) still hold and that

$$\delta_r = 0 \quad (7.3a)$$

Thus, the nonzero resonant terms in the solvability conditions become

$$E_j = -f_{jk} \bar{A}_k \exp(i\rho T_1) \quad (7.3b)$$

$$E_r = -f_{rk} A_k \exp[i(\rho - \sigma)T_1] \quad (7.3c)$$

and

$$I_r = \Gamma_{rkkk} A_k^3 \exp(-i\sigma T_1) \quad (7.3d)$$

Substituting equations (7.3b) - (7.3d) into equations (7.2), introducing the polar form (3.1), and separating the result into real and imaginary parts, we obtain

$$\omega_j(a'_j + C_j a_j) + \frac{f_{jk} a_k}{2} \sin \beta_1 + \left( \frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda \beta_{jk}}{2} a_k \right) \sin \gamma = 0 \quad (7.4a)$$

$$\omega_j a_j \alpha'_j - \frac{f_{jk} a_k}{2} \cos \beta_1 + \left( \frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda \beta_{jk}}{2} a_k \right) \cos \gamma = 0 \quad (7.4b)$$

$$\omega_j(a'_k + C_k a_k) + \frac{a_j}{2} \sin \gamma = 0 \quad (7.4c)$$

$$\omega_j a_k \alpha'_k - \frac{a_j}{2} \cos \gamma = 0 \quad (7.4d)$$

$$\omega_r(a'_r + C_r a_r) - \frac{1}{8} \Gamma_{rkkk} a_k^3 \sin \mu + \frac{1}{2} f_{rk} a_k \sin \beta_2 = 0 \quad (7.4e)$$

$$\omega_r a_r \alpha'_r + \frac{1}{8} \Gamma_{rkkk} a_k^3 \cos \beta_2 - \frac{1}{2} f_{rk} a_k \cos \beta_2 = 0 \quad (7.4f)$$

where

$$\beta_1 = \rho T_1 - \alpha_k - \alpha_j \quad (7.4g)$$

$$\gamma = \alpha_j - \alpha_k \quad (7.4h)$$



$$\mu = -\sigma T_1 + 3\alpha_k - \alpha_r \quad (7.4i)$$

$$\beta_2 = (\rho - \sigma)T_1 + \alpha_k - \alpha_r = \beta_1 + \gamma + \mu \quad (7.4j)$$

For the steady-state solutions  $a'_j = a'_k = a'_r = 0$ , and it follows that  $a_j = a_k = a_r = 0$  are possible solutions. For non-trivial solutions

$$\beta_1 = \beta_2 = \mu' = \gamma' = 0 \quad (7.5a)$$

so that

$$\rho = \alpha'_k + \alpha'_j \quad (7.5b)$$

$$\alpha'_j = \alpha'_k \quad (7.5c)$$

and

$$\sigma = 3\alpha'_k - \alpha'_r \quad (7.5d)$$

from which

$$\alpha'_j = \alpha'_k = \frac{\rho}{2} \quad (7.5e)$$

and

$$\alpha'_r = \frac{3\rho}{2} - \sigma \quad (7.5f)$$

The steady-state equations become

$$\omega_j C_j a_j + \frac{f_{jk} a_k}{2} \sin \beta_1 + \left( \frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda_{jk} a_k}{2} \right) \sin \gamma = 0 \quad (7.6a)$$

$$\omega_j C_k a_k + \frac{a_j}{2} \sin \gamma = 0 \quad (7.6b)$$

$$\omega_r C_r a_r - \frac{1}{8} \Gamma_{rkkk} a_k^3 \sin \mu + \frac{f_{rk} a_k \sin \beta_2}{2} = 0 \quad (7.6c)$$

$$\omega_j a_j \frac{\rho}{2} - \frac{f_{jk} a_k}{2} \cos \beta_1 + \left( \frac{3}{8} \Gamma_{jkkk} a_k^3 + \frac{\Lambda_{jk} a_k}{2} \right) \cos \gamma = 0 \quad (7.6d)$$

$$\omega_j a_k \rho - a_j \cos \gamma = 0 \quad (7.6e)$$

$$\omega_r a_r \left( \frac{3\rho}{2} - \sigma \right) + \frac{1}{8} \Gamma_{rkkk} a_k^3 \cos \mu - \frac{f_{rk} a_k \cos \beta_2}{2} = 0 \quad (7.6f)$$

Equations (7.6a, b, d, e) are identical to equations (6.10) and hence can be reduced to

$$a_k^2 = \frac{4}{3\Gamma_{jkkk}} \left[ -\Lambda \beta_{jk} + \omega_j^2 (4C_k C_j - \rho^2) \pm \sqrt{f_{jk}^2 - 4\omega_j^4 \rho^2 (C_j + C_k)^2} \right] \quad (7.6g)$$

$$a_j = \omega_j a_k \sqrt{\rho^2 + 4C_k^2} \quad (7.6h)$$

Moreover,  $\beta_1$  and  $\gamma$  can be obtained from these equations. To find  $a_r$ ,  $\beta_2$  and  $\mu$ , we solve equations (7.6c,f) and (7.4j) numerically. The response is the same as that in Chapter 6 for the two modes associated with the repeated frequency. Due to the presence of an internal resonance, a third mode is drawn into the response, but it does not influence the  $j$ -th and  $k$ -th modes.

## 7.2 The Case Where $\lambda \approx \omega_r + \omega_j$ and $\omega_r \approx 3\omega_j$ .

The resonances associated with this case are

$$\lambda = \omega_r + \omega_j + \epsilon^\delta \rho \quad (7.7a)$$

and

$$\omega_r = 3\omega_j + \epsilon^\delta \sigma \quad (7.7b)$$

The solvability conditions of equations (7.2) remain valid. Starting with the formulation of the three-term expansion of Section 6.2, we note

that the parametric resonances have been accounted for and that we must attempt to include the internal resonance terms

$$I_j = (\Gamma_{jrkk} + \Gamma_{jkrk} + \Gamma_{jkk r}) A_r \bar{A}_k^2 \exp(i\epsilon^\delta \sigma T_0) \quad (7.8a)$$

and

$$I_r = \Gamma_{rkkk} A_k^3 \exp(-i\epsilon^\delta \sigma T_0) \quad (7.8b)$$

Unfortunately, both of these terms are order  $\epsilon^{3\delta_0}$  and do not enter at this level of approximation.

We thus conclude that the effects of the parametric resonance and the repeated frequency dominate the response and that the effect of the internal resonance is of higher order. To this order of approximation the response in the presence of a parametric resonance is the same regardless of whether an internal resonance is present or not.

### 7.3 The Case of No Parametric Resonance with $\omega_r \approx 3\omega_j$ .

Only an internal resonance is associated with this case and it is associated with the combination

$$\omega_r = 3\omega_j + \epsilon^\delta \sigma \quad (7.9)$$

The solvability conditions of equations (7.2) remain valid. There is no parametric resonance so the  $E_n = 0$ . Using the scaling conditions of equations (5.5) thru (5.10), (6.3b) and (7.3a), we find that the only internal resonance term that appears to first order is

$$I_r = \Gamma_{rkkk} A_k^3 \exp(-i\epsilon^\delta \sigma T_0) \quad (7.10)$$



Substituting equation (7.10) into equation (7.2), introducing the polar form (3.1), and separating the result into real and imaginary parts, we obtain equations identical to equations (7.4) when

$$f_{jk} = f_{rk} = 0 \quad (7.11)$$

The resulting steady-state equations lead to the following:

$$a_k = 2 \sqrt{\frac{4\omega_j^2 C_j C_k - \Lambda B_{jk}}{3\Gamma_{jkkk}}} \quad (7.12a)$$

$$a_j = 2\omega_j C_k a_k \quad (7.12b)$$

and

$$a_r = \frac{\frac{1}{8} \Gamma_{rkkk} a_k^3}{\sqrt{\omega_r^2 (C_r^2 + \sigma^2)}} \quad (7.12c)$$

Hence we note that equations (7.12a) and (7.12b) are the limit-cycle equations developed in Chapter 5 when no resonance was present. The modes associated with the repeated frequency are independent of the internal resonance, which only serves to draw the  $r$ -th mode into the response. Moreover, the steady-state amplitudes  $a_j$  and  $a_k$  predicted by equations (7.6g) and (7.6h) approach those given by equations (7.12a) and (7.12b) as  $f_{jk}$  vanishes. We note that, if a steady-state solution exists,  $|f_{jk}| > |\rho| 2\omega_j (C_j + C_k)$  and hence  $\rho$  must vanish as  $f_{jk}$  vanishes. Finally, we note that equations (7.6c) and (7.6f) yield equation (7.12c) as  $f_{jk}$  and  $f_{rk}$  vanish.

#### 7.4 Summary

We have examined two cases of parametric resonance in the presence of the internal resonance  $\omega_r \approx 3\omega_j$ .

a. When  $\lambda$  is near twice the repeated frequency the response for the repeated modes is the same as for the case of no internal resonance. The third mode is merely drawn into the response.

b. When  $\lambda$  is away from twice the repeated frequency the response tends to be the same regardless of the presence of an internal resonance. The effect of the parametric resonance and the repeated frequency dominate the response.

c. In the case of no parametric resonance or a parametric resonance away from twice the repeated frequency, the form of the response for the modes associated with the repeated frequency is the same as that discussed in Chapter 5 where no resonance was present. Hence the only mode affected is the mode drawn into the response by the internal resonance.

## CHAPTER EIGHT

### LATERAL VIBRATIONS OF COLUMNS

The lateral vibrations of columns are used as an example where the analysis contained in Chapters, 2, 3, and 4 can be applied. The natural frequencies of oscillations are distinct. The governing equation will be nondimensionalized and converted to the form discussed in Chapters 2, 3, and 4.

#### 8.1 General

Relatively large-amplitude lateral deflections cause significant stretching of the neutral axis when the longitudinal displacements of the ends are restrained. For small, but finite deflections, one needs to use nonlinear geometrical relationships to account for this stretching. The equations governing the deflection can be written in the form

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = [p(t) + \frac{EA}{2l} \int_0^l \left(\frac{\partial w}{\partial \xi}\right)^2 d\xi] \frac{\partial^2 w}{\partial x^2} - c \frac{\partial w}{\partial t} \quad (8.1)$$

The cubic term accounts for the stretching. The terms which account for shear deformation and rotary inertia are formally the same order as the term which accounts for stretching. However, these terms are linear, and consequently, they are responsible for only small perturbations of the solution of the linear problem. Here the focus is on nonlinear effects.

It is convenient to introduce nondimensional variables (denoted by primes); thus, we let



$$x = Lx', \quad w = \frac{r^2}{L} w', \quad t = \frac{L^2}{r} \sqrt{\frac{\rho}{E}} t' \quad (8.2a)$$

$$p(t) = EA \left(\frac{r}{L}\right)^4 p'(t'), \quad C = \frac{2Ar^3}{L^2} \sqrt{\rho E} C', \quad \epsilon = \frac{r^4}{L^4} EA \quad (8.2b)$$

Substituting equation (8.2) into equation (8.1) and dropping the primes, we obtain

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = \epsilon \left\{ [p(t) + \frac{1}{4} \int_0^2 \left( \frac{\partial w(\xi, t)}{\partial \xi} \right)^2 d\xi] \frac{\partial^2 w}{\partial x^2} - 2C \frac{\partial w}{\partial t} \right\} \quad (8.3)$$

Here the characteristic length is one-half the actual length of the beam. We seek approximate solutions of equation (8.3) which are valid for small, but finite, values of  $\epsilon$ .

We express the deflection as an expansion in terms of the linear free-oscillation modes as

$$w(x, t) = \sum_{m=1}^{\infty} U_m(t; \epsilon) \phi_m(x) \quad (8.4)$$

In equation (8.4), the  $\phi_m$  are the orthonormal solutions of the following:

$$\phi_m^{iv} - \omega_m^2 \phi_m = 0 \quad (8.5a)$$

where

$$\phi_m = 0 \text{ and } \phi_m' = 0 \quad (8.5b)$$

at clamped ends and

$$\phi_m = 0 \text{ and } \phi_m'' = 0 \quad (8.5c)$$

at hinged ends. The  $\omega_m$  are the natural frequencies and the  $\phi_m$  are the linear free-oscillation modes.

Substituting equation (8.4) into equation (8.3), multiplying by  $\phi_n$ , and integrating the result from  $x = 0$  to  $x = 2$ , we obtain

$$\ddot{U}_n + \omega_n^2 U_n = \varepsilon [-2 \cos \lambda t \sum_{m=1}^{\infty} f_{nm} U_m - 2 C_n \dot{U}_n + \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \Gamma_{nmpq} U_m U_p U_q]$$

for  $n = 1, 2, \dots$  (8.6a)

where

$$P(t) = F \cos \lambda t$$

$$2f_{nm} = -F \int_0^2 \phi_n \phi_m'' dx = F \int_0^2 \phi_n' \phi_m' dx \quad (8.6b)$$

and

$$\Gamma_{nmpq} = -\frac{1}{4} \int_0^2 \phi_n' \phi_m' dx \int_0^2 \phi_p' \phi_q' dx \quad (8.6c)$$

Equations (8.6a) and (2.1) are the same.

## 8.2 Hinged-Hinged Beam

In this case,

$$\phi_n = \sin \sqrt{\omega_n} x \quad (8.7a)$$

and

$$\omega_n = \frac{1}{4} n^2 \pi^2 \quad (8.7b)$$

It follows that resonant combinations of natural frequencies abound; however, for this beam

$$\Gamma_{nmpq} = -\frac{1}{4} \omega_n \omega_p \delta_{nm} \delta_{pq} \quad (8.8a)$$

Thus all the  $I_n$  are zero. Essentially, there is no internal resonance for this beam.

From equation (8.6b) it follows that

$$f_{nm} = \frac{1}{2} \omega_n F \delta_{nm} \quad (8.8b)$$

Therefore, resonances of the combination type,  $\lambda$  near  $\omega_n \pm \omega_m$ , are not possible for this beam.

Using equations (8.7) and considering  $\lambda$  to be near  $2\omega_n$ , we obtain the following from equations (3.7):

$$a_n' + C_n a_n + \frac{1}{4} F a_n \sin \mu = 0 \quad (8.9a)$$

$$a_n \alpha_n' - 3a_n^3 \omega_n / 32 - \frac{1}{4} F a_n \cos \mu = 0 \quad (8.9b)$$

where

$$\mu = \rho T_1 - 2\alpha_n \quad (8.9c)$$

For the non-trivial steady-state solution, we find that

$$\sin \mu = \frac{4C_n}{F}, \quad \cos \mu = \frac{16\rho - 3\omega_n a_n^2}{8F} \quad (8.10a)$$

and

$$a_n = \sqrt{\frac{16\rho}{3\omega_n} \pm \frac{8}{3\omega_n} \sqrt{F^2 - 16C_n^2}} \quad (8.10b)$$

It follows from (8.10b) that

$$F \geq 4C_n \quad (8.11)$$

if a real solution for  $a_n$  can exist. If  $F \geq 4C_n$ , then

(1) when

$$\rho < -\frac{1}{2} \sqrt{F^2 - 16C_n^2}, \quad (8.12a)$$

no real solution exists;

(2) when

$$-\frac{1}{2} \sqrt{F^2 - 16C_n^2} \leq \rho \leq \frac{1}{2} \sqrt{F^2 - 16C_n^2} \quad (8.12b)$$



one real solution exists;

(3) when

$$\rho > \frac{1}{2} \sqrt{F^2 - 16C_n^2} \quad (8.12c)$$

two real solutions exist.

Now we consider the stability of the various possibilities. Using the analysis in Chapter 3, equations (3.8) through (3.13), we find that for this case equation (3.11) becomes

$$\Omega = -C_n \pm \sqrt{C_n^2 \pm \frac{3\omega_n}{32} a_{ns}^2 \sqrt{F^2 - 4C_n^2}} \quad (8.13)$$

When the negative sign inside the radical is used, the real parts of both values of  $\Omega$  are negative and hence the corresponding motion is stable. The negative sign corresponds to the larger  $a_n$  given by equation (8.10b). Hence, when two real solutions exist, only the one with the large amplitude is stable, and when only one real solution exists, it is stable.

The stability of the trivial solutions can be examined in a similar fashion. It turns out that, when  $\rho$  is in the region where there is no real solution of equation (8.10b), the trivial solution is stable; when  $\rho$  is in the region where there is one real solution of equation (8.10b), the trivial solution is unstable; and when  $\rho$  is in the region where there are two real solutions of equation (8.10b), the trivial solution is stable.

The results are illustrated in Figures 1-3 when  $\lambda$  is near  $\omega_1$ . For combinations of  $F$  and  $\rho$  in Region I, no nontrivial steady-state solution

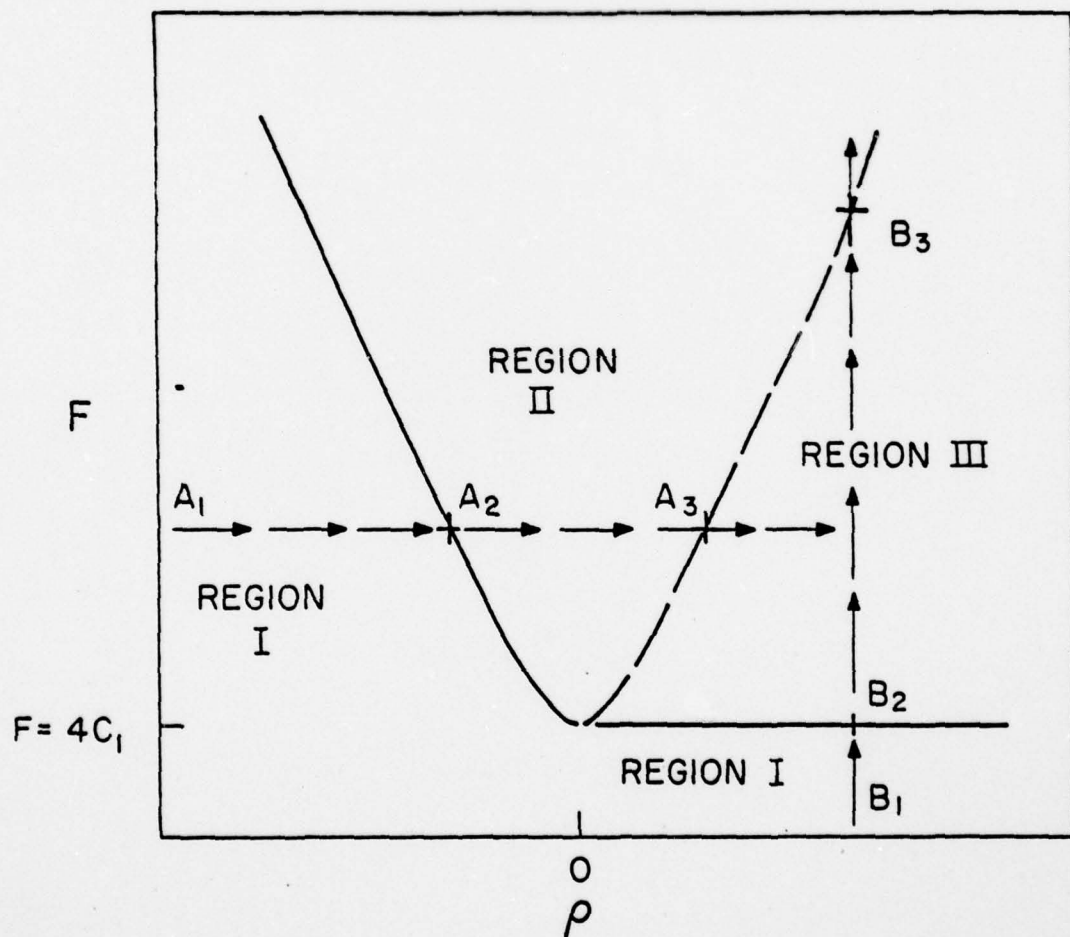


Figure 1. Parameter space,  $F$  is the amplitude and  $\rho(\epsilon\rho = \lambda - 2\omega_1)$  is the frequency of the excitation.

exists, and the trivial solution is stable. In Region II, the trivial solution is unstable, while the only nontrivial solution is stable. In Region III, two nontrivial solutions are possible. The trivial solution and the larger-amplitude nontrivial solution are stable, while the smaller-amplitude nontrivial solution is unstable.

In Region I, it appears that regardless of how large the initial amplitude is, the phasing never becomes such that the rate at which work is being done is as large as the rate at which energy is being dissipated. Indeed, if the damping term were deleted from equation (8.9a), the predicted response would still decay. Thus, we conclude that, for the linear as well as the nonlinear system, the phasing is such that the force actually does negative work and contributes to the decay.

In Region II, for a linear system the phasing, which does not change with amplitude, is such that work is being done at a faster rate than energy is being dissipated and thus the response to any initial disturbance grows without bound. For the nonlinear system, the phasing for large amplitudes differs from the phasing in the linear system as a consequence of the nonlinear term in equation (8.9a). The effect is to limit the rate at which work is being done to the rate at which energy is being dissipated and thereby to produce a bounded harmonic response. If the initial disturbance is very large, the response will decay until the steady-state solution is reached, while if the initial disturbance is very small the response will grow (the system being governed by the linear equations when the amplitude is small) until the nonlinear term in equation (8.9b) becomes large enough to cause the phase to shift



significantly. Thus, in Region II all initial disturbances produce the same steady-state response.

In Region III, the response of the linear system to an initial disturbance always decays. The mechanism causing the decay is the same as in Region I. The results for the nonlinear system show that only the larger of the two possible steady-state responses is stable. Thus, it appears that, if the initial disturbance is small, the nonlinear term in equation (8.9b) does not have a strong influence on the resulting motion and the system behaves essentially as a linear system; the motion decays. On the other hand, when the initial disturbance is large enough to produce the nontrivial steady-state response, the nonlinear term has a strong influence, and changes in the phase occur. Thus, in Region III there is the possibility of producing motions which have characteristics that are similar to those of the motions in Region I as well as those in Region II. The type of motion is determined by the amplitude and phase of the initial disturbance. This is a rare example in which a nontrivial, steady-state response of the nonlinear system exists in a region where the response of the linear system decays.

Let us suppose that the frequency of the excitation is varied while the amplitude is held fixed. This process is represented by the line through points  $A_1$ ,  $A_2$  and  $A_3$ . Between points  $A_1$  and  $A_2$ , the trivial solution is stable; moreover, it is the only steady-state solution possible. Between points  $A_2$  and  $A_3$ , the trivial solution is unstable, and the only realizable solution is the one given by equation (8.10b). Beyond point  $A_3$ , the trivial solution is again stable and two solutions

are realizable -- the trivial solution and the larger of the two solutions given by equation (8.10b).

Finally, let us suppose the amplitude of the excitation is varied while the frequency is held fixed. This process is represented by the line through points  $B_1$ ,  $B_2$  and  $B_3$ . Between points  $B_1$  and  $B_2$ , only the trivial solution is possible because the requirement of equation (8.11) is not satisfied. Between points  $B_2$  and  $B_3$ , two solutions are realizable-- the trivial solution, which is stable, and the larger of the two solutions given by equation (8.10b). Beyond point  $B_3$ , the trivial solution is unstable and the only realizable solution is the one given by equation (8.10b).

For the first process, the amplitude of the solution is plotted as a function of  $\rho$  in Figure 2. If the frequency of the excitation decreases from a large value, we note that upon reaching point  $A_3$ , where the trivial solution becomes unstable, there is a jump in the value of  $a_1$  up from point  $A_3$  to point  $A_3'$ . This process is indicated by the arrows.

For the second process, the amplitude of the solution is plotted as a function of  $F$  in Figure 3. We note the presence of a jump phenomenon. If  $F$  is increased slowly from Point 0 toward point  $B_1$ ,  $a_1$  remains zero. However, the trivial solution becomes unstable at  $B_3$  and a slight increase in  $F$  at this point causes  $a_1$  to jump up to point  $B_3'$ . Then further increases cause  $a_1$  to follow curve  $CB_3'D$  toward point D. When  $F$  is decreased  $a_1$  follows curve  $CB_3'D$  away from point D pasts point  $B_3'$ , without jumping down to point  $B_3$ , until it reaches point C. At this point a slight decrease in  $F$  causes  $a_1$  to jump back to zero at point  $B_2$ .

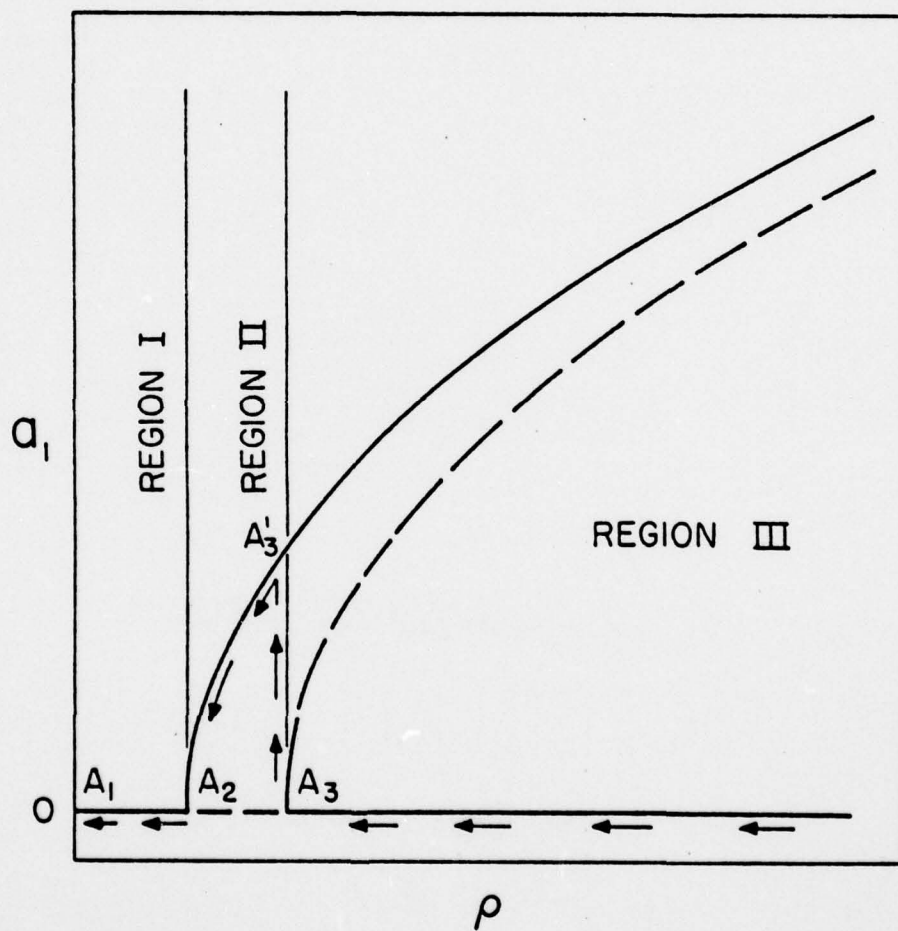


Figure 2. Amplitude of the response,  $a_1$ , as a function of the frequency of the excitation,  $\rho(\epsilon\rho = \lambda - 2\omega_1)$  for constant amplitude of the excitation,  $F$ . — stable, ---- unstable.



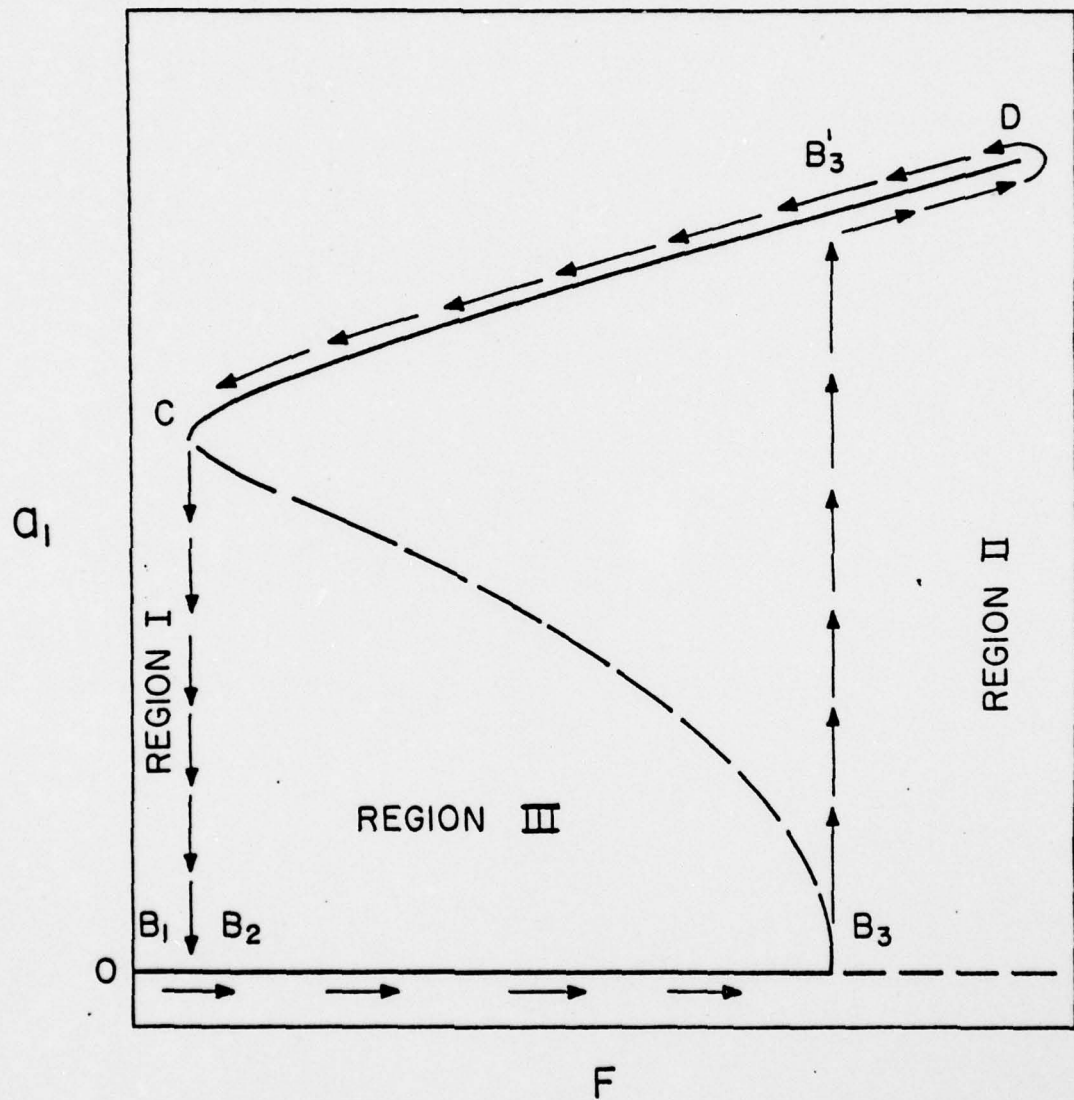


Figure 3. Amplitude of the response,  $a_1$ , as a function of the amplitude of the excitation,  $F$ , for constant frequency of the excitation,  $\rho$ . — stable, ---- unstable.

The arrows indicate this path. Figure 3 corresponds to  $\rho$  greater than zero. As  $\rho$  decreases, point C approaches point  $B_3$ , and the multi-valued region and hence the jump vanish.

The results for  $\lambda$  near  $2\omega_n$ , where  $n = 2, 3, \dots$ , are similar to the above results.

Next we consider an example for which the internal resonance plays an important role.

### 8.3 Hinged-Clamped Beam

For a beam which is hinged at the left end and clamped at the right

$$\phi_n = \psi_n \left[ \sin \sqrt{\omega_n} x - \frac{\sin 2\sqrt{\omega_n}}{\sinh 2\sqrt{\omega_n}} \sinh \sqrt{\omega_n} x \right] \quad (8.16a)$$

where

$$\psi_n = \frac{\sinh 2\sqrt{\omega_n}}{[\sinh^2 2\sqrt{\omega_n} - \sin^2 2\sqrt{\omega_n}]^{1/2}} \quad (8.16b)$$

and the  $\omega_n$  are the roots of

$$\tanh 2\sqrt{\omega_n} - \tan 2\sqrt{\omega_n} = 0 \quad (8.16c)$$

The first five roots of equation (8.16c) are

$$\omega_1 = 3.855, \omega_2 = 12.491, \omega_3 = 26.062, \omega_4 = 44.567, \text{ and}$$

$$\omega_5 = 68.008$$

Referring to equation (2.8a), we find that for the first five modes the combinations for which  $\epsilon\sigma$  is below eight percent of  $\omega_n$  are

$$3\omega_1 = \omega_2 - \epsilon\sigma_1 \quad (8.18a)$$

$$\omega_1 + \omega_2 + \omega_3 = \omega_4 + \epsilon\sigma_2 \quad (8.18b)$$

$$2\omega_2 + \omega_4 = \omega_5 + \epsilon\sigma_3 \quad (8.18c)$$

$$\omega_2 + 2\omega_3 = \omega_5 + \epsilon\sigma_4 \quad (8.18d)$$

$$\omega_4 + \omega_3 - \omega_1 = \omega_5 + \epsilon\sigma_5 \quad (8.18e)$$

$$2\omega_4 - \omega_3 = \omega_5 + \epsilon\sigma_6 \quad (8.18f)$$

Referring to equations (2.9), we find that the conditions for the elimination of secular terms are

$$\begin{aligned} \omega_1(a_1' + C_1a_1) - 3Q_1a_1^2a_2\sin\mu_1 + Q_2a_2a_3a_4\sin\mu_2 - Q_5a_3a_4a_5\sin\mu_5 \\ - \text{Imag}[E_1\exp(-i\alpha_1)] = 0 \end{aligned} \quad (8.19)$$

$$\begin{aligned} \omega_1a_1\alpha_1' + \frac{1}{8}a_1 \sum_{m=1}^5 \gamma_{1m}a_m^2 + 3Q_1a_1^2a_2\cos\mu_1 + Q_2a_2a_3a_4\cos\mu_2 \\ + Q_5a_3a_4a_5\cos\mu_5 + \text{Re}[E_1\exp(-i\alpha_1)] = 0 \end{aligned} \quad (8.19b)$$

$$\begin{aligned} \omega_2(a_2' + C_2a_2) + Q_1a_1^3\sin\mu_1 + Q_2a_1a_3a_4\sin\mu_2 + 2Q_3a_2a_4a_5\sin\mu_3 \\ + Q_4a_3^2a_5\sin\mu_4 - \text{Imag}[E_2\exp(-i\alpha_2)] = 0 \end{aligned} \quad (8.20a)$$

$$\begin{aligned} \omega_2a_2\alpha_2' + \frac{1}{8}a_2 \sum_{m=1}^5 \gamma_{2m}a_m^2 + Q_1a_1^3\cos\mu_1 + Q_2a_1a_3a_4\cos\mu_2 \\ + 2Q_3a_2a_4a_5\sin\mu_3 + Q_4a_3^2a_5\sin\mu_4 + \text{Re}[E_2\exp(-i\alpha_2)] = 0 \end{aligned} \quad (8.20b)$$

$$\begin{aligned} \omega_3(a_3' + C_3a_3) + Q_2a_1a_2a_4\sin\mu_2 + 2Q_4a_2a_3a_5\sin\mu_4 + Q_5a_1a_4a_5\sin\mu_5 \\ - Q_6a_4^2a_5\sin\mu_6 - \text{Imag}[E_3\exp(-i\alpha_3)] = 0 \end{aligned} \quad (8.21a)$$



$$\begin{aligned} \omega_3 a_3 \alpha_3^1 + \frac{1}{8} a_3 \sum_{m=1}^5 \gamma_{3m} a_m^2 + Q_2 a_1 a_2 a_4 \cos \mu_2 + 2Q_4 a_2 a_3 a_5 \cos \mu_4 \\ + Q_5 a_1 a_4 a_5 \cos \mu_5 + Q_6 a_4^2 a_5 \cos \mu_6 + \operatorname{Re}[E_3 \exp(-i\alpha_3)] = 0 \end{aligned} \quad (8.21b)$$

$$\begin{aligned} \omega_4 (a_4^1 + C_4 a_4) - Q_2 a_1 a_2 a_3 \sin \mu_2 + Q_3 a_2^2 a_5 \sin \mu_3 + Q_5 a_1 a_3 a_5 \sin \mu_5 \\ + 2Q_6 a_3 a_4 a_5 \sin \mu_6 - \operatorname{Imag}[E_4 \exp(-i\alpha_4)] = 0 \end{aligned} \quad (8.22a)$$

$$\begin{aligned} \omega_4 a_4 \alpha_4^1 + \frac{1}{8} a_4 \sum_{m=1}^5 \gamma_{4m} a_m^2 + Q_2 a_1 a_2 a_3 \cos \mu_2 + Q_3 a_2^2 a_5 \cos \mu_3 \\ + Q_5 a_1 a_3 a_5 \cos \mu_5 + 2Q_6 a_3 a_4 a_5 \cos \mu_6 + \operatorname{Re}[E_4 \exp(-i\alpha_4)] = 0 \end{aligned} \quad (8.22b)$$

$$\begin{aligned} \omega_5 (a_5^1 + C_5 a_5) - Q_3 a_2^2 a_4 \sin \mu_3 - Q_4 a_2 a_3^2 \sin \mu_4 - Q_5 a_1 a_3 a_4 \sin \mu_5 \\ - Q_6 a_3 a_4^2 \sin \mu_6 - \operatorname{Imag}[E_5 \exp(-i\alpha_5)] = 0 \end{aligned} \quad (8.23a)$$

$$\begin{aligned} \omega_5 a_5 \alpha_5^1 + \frac{1}{8} a_5 \sum_{m=1}^5 \gamma_{5m} a_m^2 + Q_3 a_2^2 a_4 \cos \mu_3 + Q_6 a_2 a_3^2 \sin \mu_4 \\ + Q_5 a_1 a_3 a_4 \cos \mu_5 + Q_6 a_3 a_4^2 \cos \mu_6 + \operatorname{Re}[E_5 \exp(-i\alpha_5)] = 0 \end{aligned} \quad (8.23b)$$

where

$$\begin{aligned} Q_1 &= \Gamma_{1112}/8 \\ Q_2 &= (\Gamma_{1423} + \Gamma_{1234} + \Gamma_{1324})/4 \\ Q_3 &= (\Gamma_{2254} + 2\Gamma_{2425})/8 \\ Q_4 &= (\Gamma_{2533} + 2\Gamma_{2335})/8 \\ Q_5 &= (\Gamma_{1435} + \Gamma_{1345} + \Gamma_{1534})/4 \\ Q_6 &= (\Gamma_{3544} + 2\Gamma_{3445})/8 \end{aligned}$$

$$\mu_1 = \alpha_2 - 3\alpha_1 + \sigma_1 T_1$$

$$\mu_2 = \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \sigma_2 T_1$$

$$\mu_3 = 2\alpha_2 + \alpha_4 - \alpha_5 + \sigma_3 T_1$$

$$\mu_4 = \alpha_2 + 2\alpha_3 - \alpha_5 + \sigma_4 T_1$$

$$\mu_5 = \alpha_3 + \alpha_4 - \alpha_1 - \alpha_5 - \sigma_5 T_1$$

$$\mu_6 = 2\alpha_4 - \alpha_3 - \alpha_5 + \sigma_6 T_1$$

and the  $E_n$  depend on the excitation.

In this section, the following three cases are considered:

- (1)  $\lambda$  near  $2\omega_1$
- (2)  $\lambda$  near  $2\omega_2$
- (3)  $\lambda$  near  $\omega_1 + \omega_2$

It follows that the  $E_n$ , for  $n = 3, 4$ , and  $5$ , are zero for all three cases being considered; thus, from equations (8.19) - (8.23), it follows that a possible solution is

$$a_3 = a_4 = a_5 \equiv 0$$

and  $a_1$  and  $a_2$  are given by

$$\omega_1(a_1' + C_1 a_1) - 3Q_1 a_1^2 a_2 \sin \mu_1 - \text{Imag}[E_1 \exp(-i\alpha_1)] = 0 \quad (8.24a)$$

$$\omega_1 a_1 \alpha_1' + \frac{1}{8} a_1 (\gamma_{11} a_1^2 + \gamma_{12} a_2^2) + 3Q_1 a_1^2 a_2 \cos \mu_1 + \text{Re}[E_1 \exp(-i\alpha_1)] = 0 \quad (8.24b)$$

and

$$\omega_2(a_2' + C_2 a_2) + Q_1 a_1^3 \sin \mu_1 - \text{Imag}[E_2 \exp(-i\alpha_2)] = 0 \quad (8.25a)$$

$$\omega_2 a_2 \alpha_2' + \frac{1}{8} a_2 (\gamma_{21} a_1^2 + \gamma_{22} a_2^2) + Q_1 a_1^3 \cos \mu_1 + \text{Re}[E_2 \exp(-i\alpha_2)] = 0 \quad (8.25b)$$

which are either equations (4.30) or (4.45) depending on the form of the parametric resonance. Tso and Asmis considered the system above, but took  $Q_1$  to be zero.

Case (1),  $\lambda$  near  $2\omega_1$

To express the nearness of  $\lambda$  to  $2\omega_1$ , we introduce another detuning factor  $\rho$  defined by

$$\lambda = 2\omega_1 + \epsilon\rho = \omega_2 - \omega_1 + \epsilon(\rho - \sigma_1) \quad (8.26)$$

where equation (8.18a) was used. Then equations (8.24) and (8.25) can be rewritten exactly as (4.30) with  $m=1$  and  $k=2$ .

$$\begin{aligned} \omega_1(a_1 + C_1 a_1) + \frac{1}{2} f_{11} a_1 \sin \beta - \frac{1}{2} f_{12} a_2 \sin(\beta - \mu_1) \\ - 3Q_1 a_1^2 a_2 \sin \mu_1 = 0 \end{aligned} \quad (8.27a)$$

$$\begin{aligned} \omega_1 a_1 \alpha_1 + \frac{1}{8} a_1 (\gamma_{11} a_1^2 + \gamma_{12} a_2^2) - \frac{1}{2} f_{11} a_1 \cos \beta - \frac{1}{2} f_{12} a_2 \cos(\beta - \mu_1) \\ + 3Q_1 a_1^2 a_2 \cos \mu_1 = 0 \end{aligned} \quad (8.27b)$$

$$\omega_2(a_2 + C_2 a_2) + \frac{1}{2} f_{21} a_1 \sin(\beta - \mu_1) + Q_1 a_1^3 \sin \mu_1 = 0 \quad (8.27c)$$

$$\begin{aligned} \omega_2 a_2 \alpha_2 + \frac{1}{8} a_2 (\gamma_{21} a_1^2 + \gamma_{22} a_2^2) - \frac{1}{2} f_{21} a_1 \cos(\beta - \mu_1) + Q_1 a_1^3 \cos \mu_1 = 0 \\ (8.27d) \end{aligned}$$

where

$$\beta = \rho T_1 - 2\alpha_1 \quad (8.28a)$$

$$\mu_1 = \sigma_1 T_1 + \alpha_2 - 3\alpha_1 \quad (8.28b)$$

For the steady-state response,

$$a_1' = a_2' = 0 \quad \text{and} \quad \mu_1' = \beta' = 0 \quad (8.29)$$



It follows that the trivial solution is possible:

$$a_1 = a_2 = 0 \quad (8.30)$$

If neither  $a_1$  nor  $a_2$  is zero, one can use equations (8.28) and (8.29) to rewrite equations (8.27) as

$$\omega_1 C_1 a_1 + \frac{1}{2} f_{11} a_1 \sin \beta - \frac{1}{2} f_{12} a_2 \sin(\beta - \mu_1) - 3Q_1 a_1^2 a_2 \sin \mu_1 = 0 \quad (8.31a)$$

$$\begin{aligned} \frac{1}{2} \omega_1 a_1 \rho + \frac{1}{8} (\gamma_{11} a_1^2 + \gamma_{12} a_2^2) a_1 - \frac{1}{2} f_{11} a_1 \cos \beta - \frac{1}{2} f_{12} a_2 \cos(\beta - \mu_1) \\ + 3Q_1 a_1^2 a_2 \cos \mu_1 = 0 \end{aligned} \quad (8.31b)$$

$$\omega_2 C_2 a_2 + \frac{1}{2} f_{21} a_1 \sin(\beta - \mu_1) + Q_1 a_1^3 \sin \mu_1 = 0 \quad (8.31c)$$

$$\begin{aligned} \omega_2 a_2 \left( \frac{3}{2} \rho - \sigma_1 \right) + \frac{1}{8} (\gamma_{21} a_1^2 + \gamma_{22} a_2^2) a_2 - \frac{1}{2} f_{21} a_1 \cos(\beta - \mu_1) \\ + Q_1 a_1^3 \cos \mu_1 = 0 \end{aligned} \quad (8.31d)$$

A Newton-Raphson technique is used to find the solutions of equations (8.31).

In Figure 4, some typical results show the amplitudes of the first and second modes ( $a_1$  and  $a_2$ ) as functions of the amplitude of the excitation  $F$ , (refer to equation 8.6b). We note the multiplicity of possible jumps; jumps are indicated by the arrows. We can trace the histories of  $a_1$  and  $a_2$  as  $F$  increases very slowly from zero to a large value and then slowly decreases back to zero. Initially, both  $a_1$  and  $a_2$  are zero, and they remain zero until point A is reached. At this point the trivial solution becomes unstable. Then the amplitudes jump either to curve BCDE or FGH. If the jump is to G, then as  $F$  continues to increase, the

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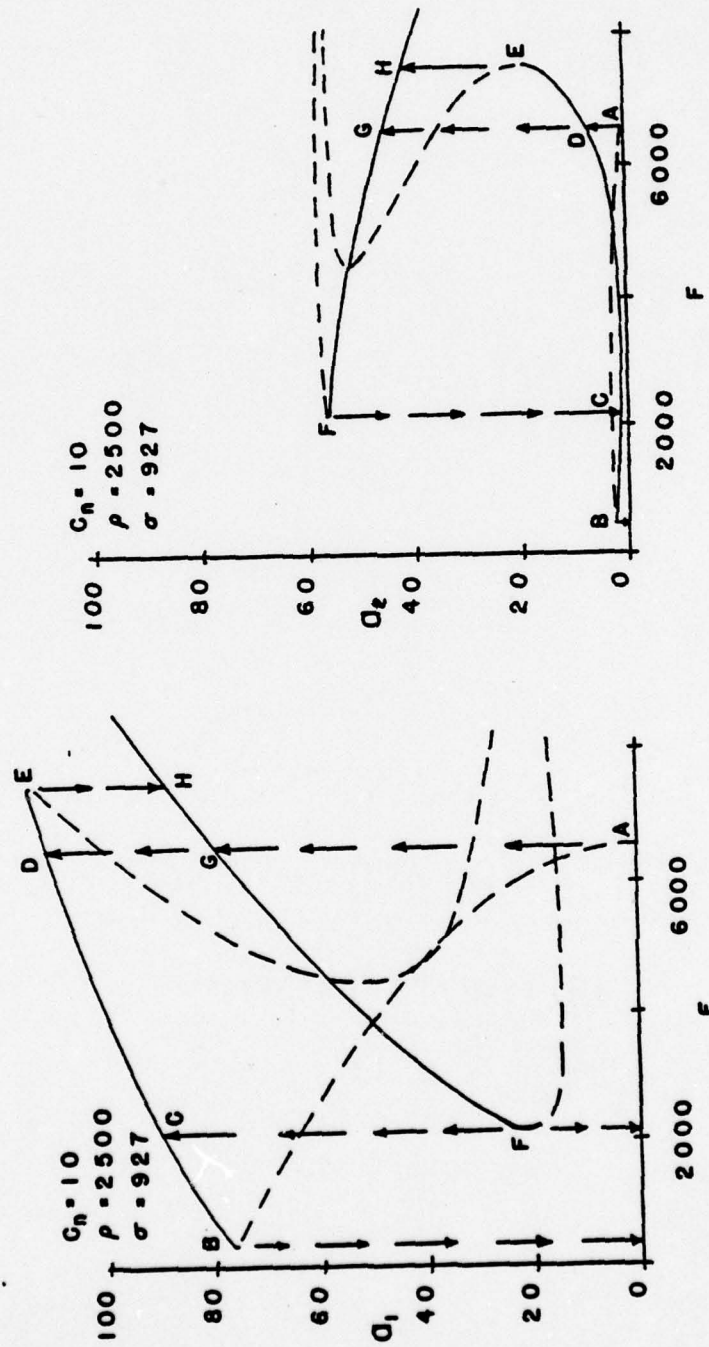


Figure 4. The modal amplitudes as functions of the amplitude of the excitation,

$F$ , for constant frequency of the excitation,  $\rho(\epsilon\rho = \lambda - 2\omega_1)$ .

— stable, ---- unstable.



response follows curve FGH. If the jump is to D, then as  $F$  continues to increase the response follows BCDE until another jump occurs from E to H. An additional increase in  $F$  causes the response to follow FGH again. Decreasing  $F$  at this time causes the amplitudes to remain on FGH until  $F$  is reached. Then a jump to either C or the trivial solution occurs. If the jump is to C, then an increase or decrease in  $F$  causes the response to follow BCDE until the next jump is encountered. The jump at E has already been discussed and the one at B returns the amplitudes to the trivial solution. In Figure 5 the amplitudes are shown as a function of the detuning  $\rho$ . Possible jumps are again indicated by the arrows. The response for  $a_1$  is initially similar to that of Figure 2 beginning with the trivial solution when  $\rho$  is very negative. As  $\rho$  increases the trivial solution becomes unstable and the amplitudes follow curve ABC as in Figure 2. As  $\rho$  increases beyond C a transfer of energy occurs between the modes, and  $a_2$  grows at the expense of  $a_1$  until E is reached. The dominance of the second mode is quite clear. An additional increase in  $\rho$  produces jumps. The amplitudes jump either to G or they jump to the trivial solution. A further increase in  $\rho$  will cause the response to continue along FG or remain trivial, depending on which way the jump occurred.

At this point, if the trivial response was attained, the solution would remain trivial for a decrease in  $\rho$  until H is reached. Here another jump occurs from H to B since the trivial solution is unstable below H. Once at B, an increase or decrease in  $\rho$  would cause the response to follow ABCDE.

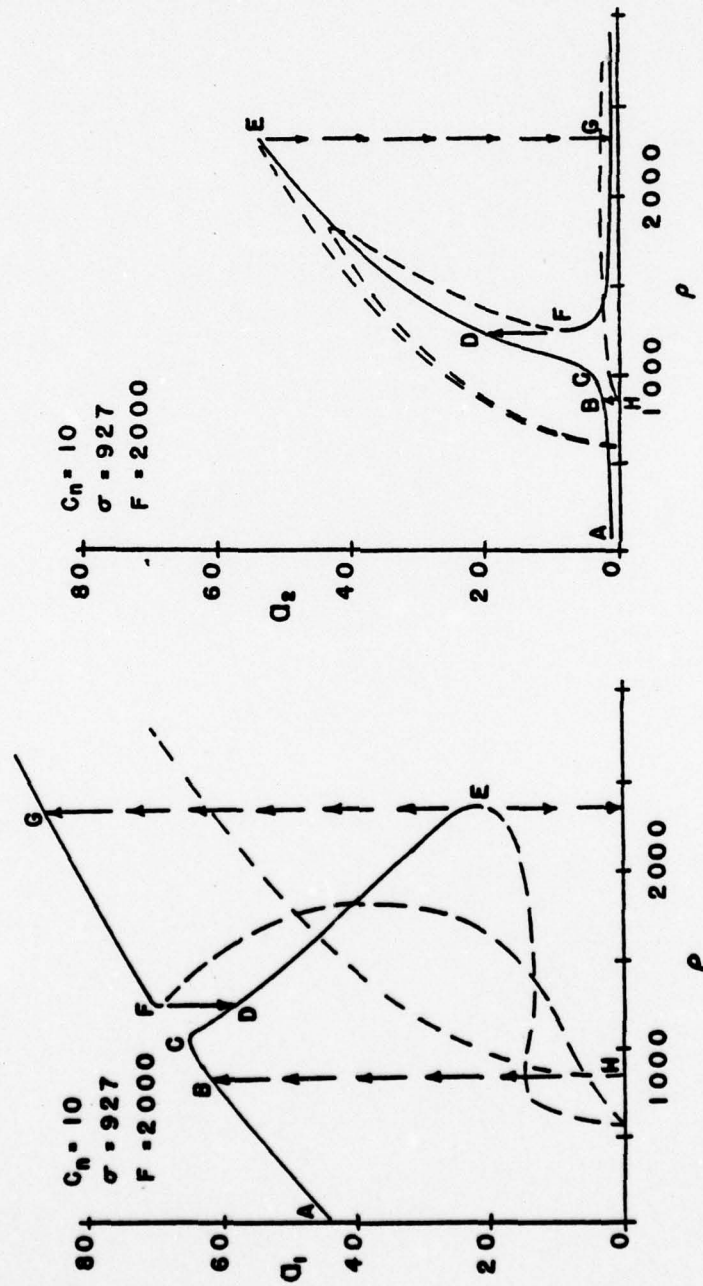


Figure 5. The modal amplitudes as functions of the frequency of the excitation,

$\rho(\epsilon\rho = \lambda - 2\omega_1)$ , for constant amplitude of the excitation,  $F$ .

— stable, ---- unstable.

If the jump from E had been to G and then  $\rho$  decreased, the response would follow GF to F where jumps again occur. Both amplitudes could become trivial or jump to D. An increase or decrease in  $\rho$  at this point would cause the response to follow ABCDE again or remain trivial until H is reached as discussed above.

Case (2),  $\lambda$  near  $2\omega_2$

In this case, we introduce the detuning parameter  $\rho$  according to

$$\lambda = 2\omega_2 + \varepsilon\rho \quad (8.32)$$

For the steady-state, equations (8.34) and (8.35) reduce to equations (4.49) with  $m = 1$  and  $k = 2$

$$a_1(\omega_1 C_1 - 3Q_1 a_1 a_2 \sin\mu_1) = 0 \quad (8.33a)$$

$$a_1[\omega_1(\sigma_1 + \frac{1}{2}\rho)/3 + \frac{1}{8}(\gamma_{11}a_1^2 + \gamma_{12}a_2^2) + 3Q_1 a_1 a_2 \cos\mu_1] = 0 \quad (8.33b)$$

$$\omega_2 C_2 a_2 + \frac{1}{2} f_{22} a_2 \sin\beta + Q_1 a_1^3 \sin\mu_1 = 0 \quad (8.34a)$$

$$\frac{1}{2} \omega_2 \rho a_2 + \frac{1}{2} (\gamma_{21} a_1^2 + \gamma_{22} a_2^2) a_2 - \frac{1}{2} f_{22} a_2 \cos\beta + Q_1 a_1^3 \cos\mu_1 = 0 \quad (8.34b)$$

where

$$\beta = \rho T_1 - 2\alpha_2 \quad (8.35a)$$

and as above

$$\mu_1 = \sigma_1 T_1 + \alpha_2 - 3\alpha_1 \quad (8.35b)$$

From equations (8.33) and (8.34), it appears that there are two possibilities for a nontrivial solution: either  $a_1$  is zero and  $a_2$  is nonzero or both  $a_1$  and  $a_2$  are nonzero. Thus, the situation is the same as that in Case (1). When  $a_1$  is zero, the results resemble those illustrated in Figures 1-3; consequently, they are not given here.



The results when both  $a_1$  and  $a_2$  differ from zero are illustrated in Figures 6 and 7. The top and bottom branches go together, and the middle two branches go together. The important thing to note is that over a fairly wide range  $a_1$  is much larger than  $a_2$ . This means that the first mode dominates the deflection, though the second mode is the only one directly excited. Consequently, when both  $a_1$  and  $a_2$  differ from zero, there can be a significant transfer of energy from the second mode to the first.

Case (3),  $\lambda$  near  $\omega_1 + \omega_2$

In this case we introduce a detuning parameter  $\rho$  defined as

$$\lambda = 4\omega_1 + \varepsilon\rho = \omega_2 + \omega_1 + \varepsilon(\rho - \sigma_1) \quad (8.36)$$

Also, we let

$$\mu_1 = \sigma_1 T_1 + \alpha_2 - 3\alpha_1 \quad (8.37a)$$

$$\mu_2 = (\rho - \sigma_1)T_1 - \alpha_1 - \alpha_2 \quad (8.37b)$$

For steady-state solutions  $a'_n = \mu'_n = 0$ , and equations (8.24) and (8.25) yield those of (4.22) with  $m = 1$  and  $k = 2$ ; that is

$$\omega_1 C_1 a_1 + \frac{f_{12} a_2}{2} \sin \mu_2 - 3Q_1 a_1^2 a_2 \sin \mu_1 = 0 \quad (8.38a)$$

$$\omega_1 a_1 \frac{\rho}{4} + \frac{1}{8} (\gamma_{11} a_1^2 + \gamma_{12} a_2^2) a_1 - \frac{f_{12} a_2}{2} \cos \mu_2 + 3Q_1 a_1^2 a_2 \cos \mu_1 = 0 \quad (8.38b)$$

$$\omega_2 C_2 a_2 + \frac{f_{21} a_1}{2} \sin \mu_2 + Q_1 a_1^3 \sin \mu_1 = 0 \quad (8.39a)$$

$$\omega_2 a_2 \left( \frac{3\rho}{4} - \sigma_1 \right) + \frac{1}{8} (\gamma_{12} a_1^2 + \gamma_{22} a_2^2) a_2 - \frac{f_{21} a_1}{2} \cos \mu_2 + Q_1 a_1^3 \cos \mu_1 = 0 \quad (8.39b)$$

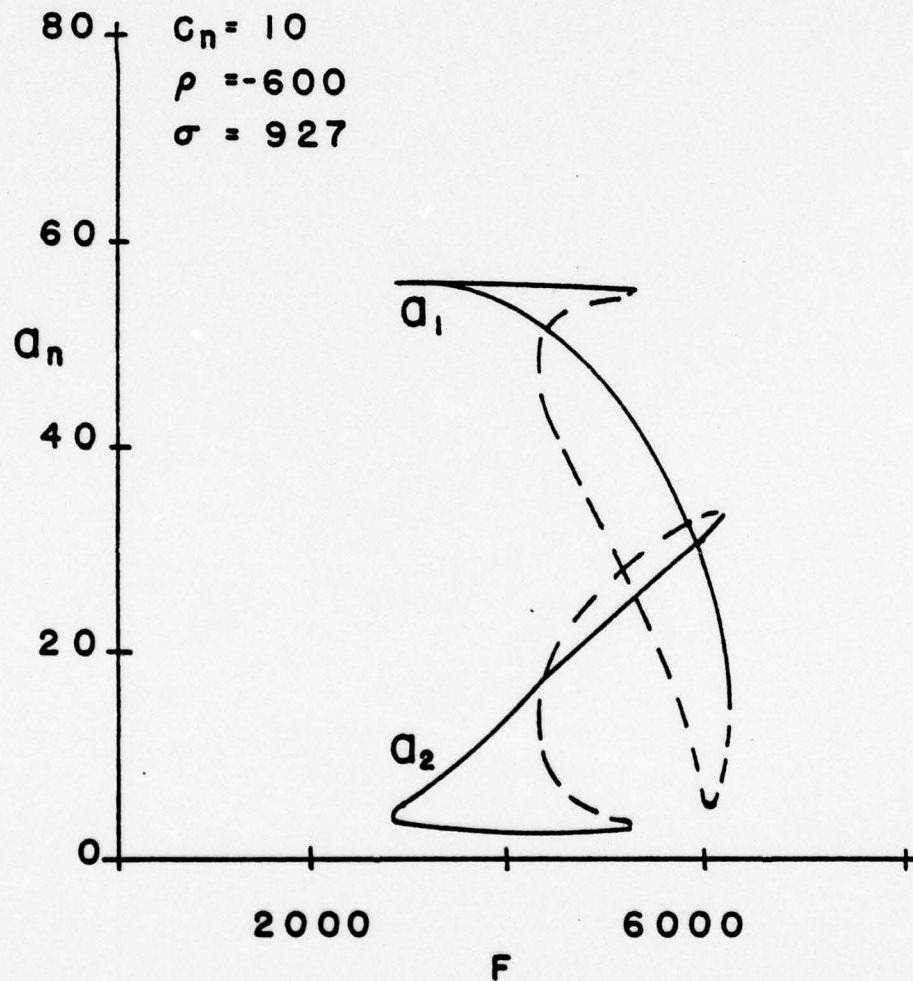


Figure 6. The modal amplitudes as functions of the amplitude of the excitation  $F$ , for constant frequency of the excitation,  $\rho(\epsilon\rho = \lambda - 2\omega_2)$ .  
 — stable, ---- unstable.

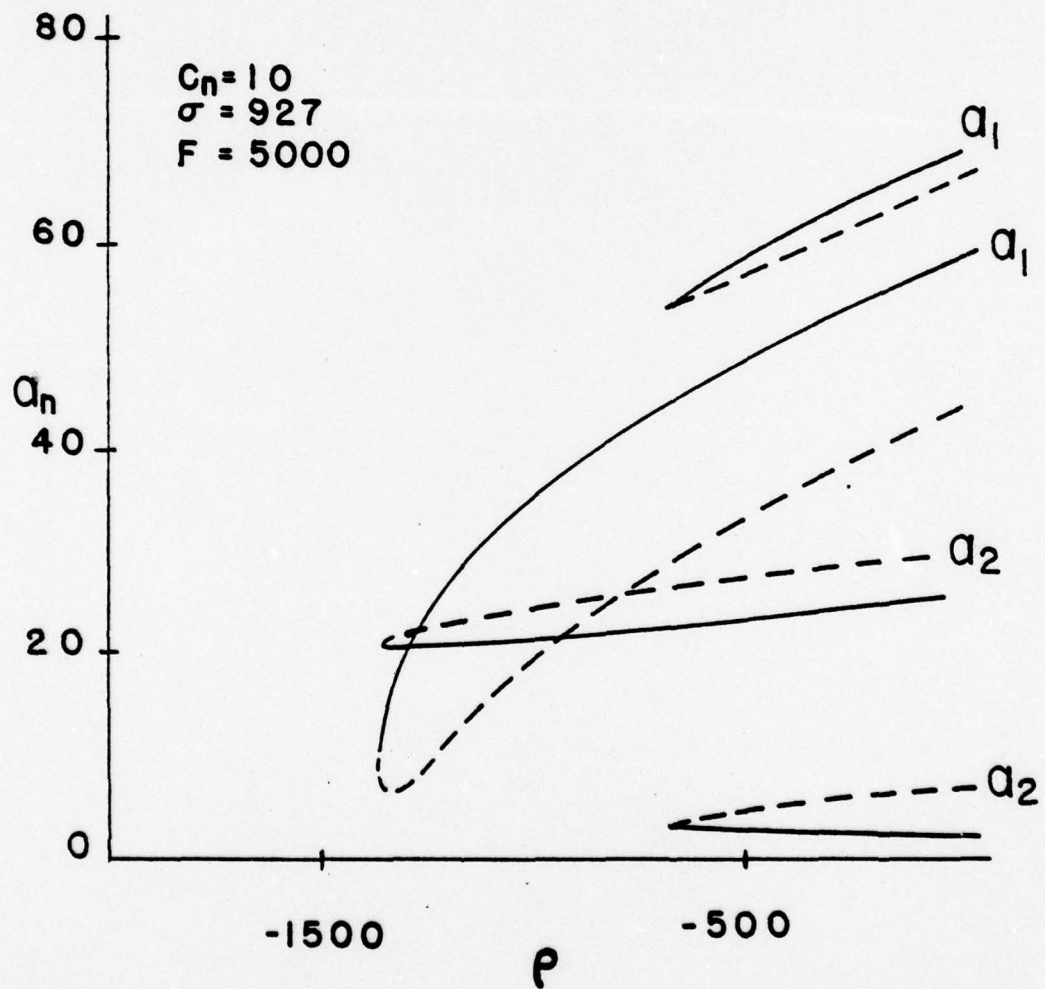


Figure 7. The modal amplitudes as functions of the frequency of the excitation,  $\rho(\epsilon\rho = \lambda - 2\omega_2)$ , for constant amplitude of the excitation. — stable, ---- unstable.



Two possibilities exist: either  $a_1$  and  $a_2$  are zero, or neither is zero. Figures 8 and 9 show the variation of  $a_1$  and  $a_2$  with  $F$  and  $\rho$ . In both figures, the top curves for  $a_1$  go with the bottom curves for  $a_2$ , and the two middle curves go together. Again we note that over a wide range,  $a_1$  is much larger than  $a_2$ , and hence the first mode dominates the response.

#### 8.4 Observations - Internal and Parametric Resonances.

- a. The Effect of the Detuning Parameter  $\rho$  on the Response,  
 $a_n$  vs  $F$ ,  $\lambda$  near  $2\omega_1$ .

Some interesting observations may be noted concerning the behavior represented in Figure 4, which depicts the relationship between the response amplitudes,  $a_n$ , and the amplitude of the forcing function,  $F$ . The detuning parameter  $\rho$  measures the closeness of the frequency of the excitation to  $2\omega_1$ . In Figure 10, when  $\rho$  is small, the response is essentially indistinguishable from that of Figure 3, i.e. the case when no internal resonance is present. The amplitude of the second-mode is almost zero. As the frequency of the excitation increases, (i.e.,  $\rho$  increases), the effect of the internal resonance begins to appear. The response may resemble that in Figure 10, or the second mode may dominate the first mode as shown in Figures 11 and 12. As the detuning of the parametric resonance approaches that of the internal detuning ( $\rho \approx \sigma$ ), we see in Figure 11, strong modal interaction and the possible dominance of  $a_2$  over  $a_1$  for a large range of the amplitude of the excitation,  $F$ . In Figures 4 and 12, for  $\rho$  greater than  $\sigma$ , the second can dominate the

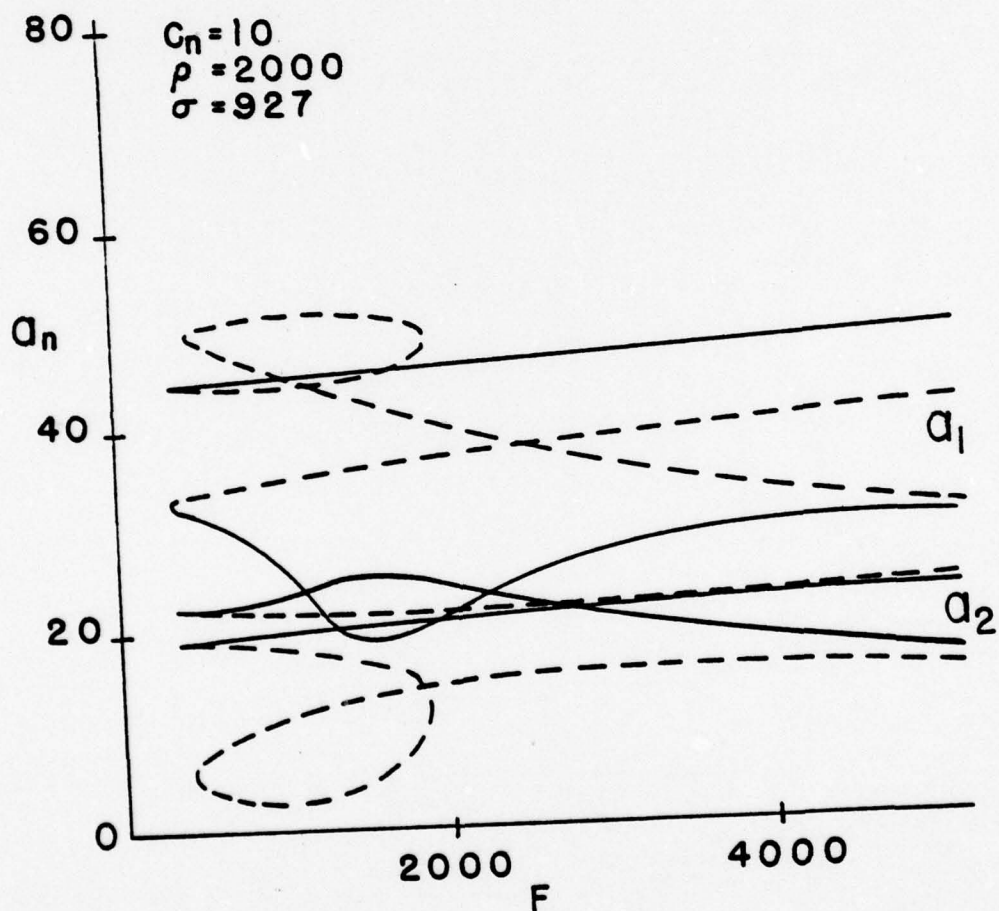


Figure 8. The modal amplitudes as functions of the amplitude of the excitation,  $F$ , for constant frequency of the excitation,  $\rho(\epsilon p = \lambda - \omega_1 - \omega_2)$ . — stable, ---- unstable.

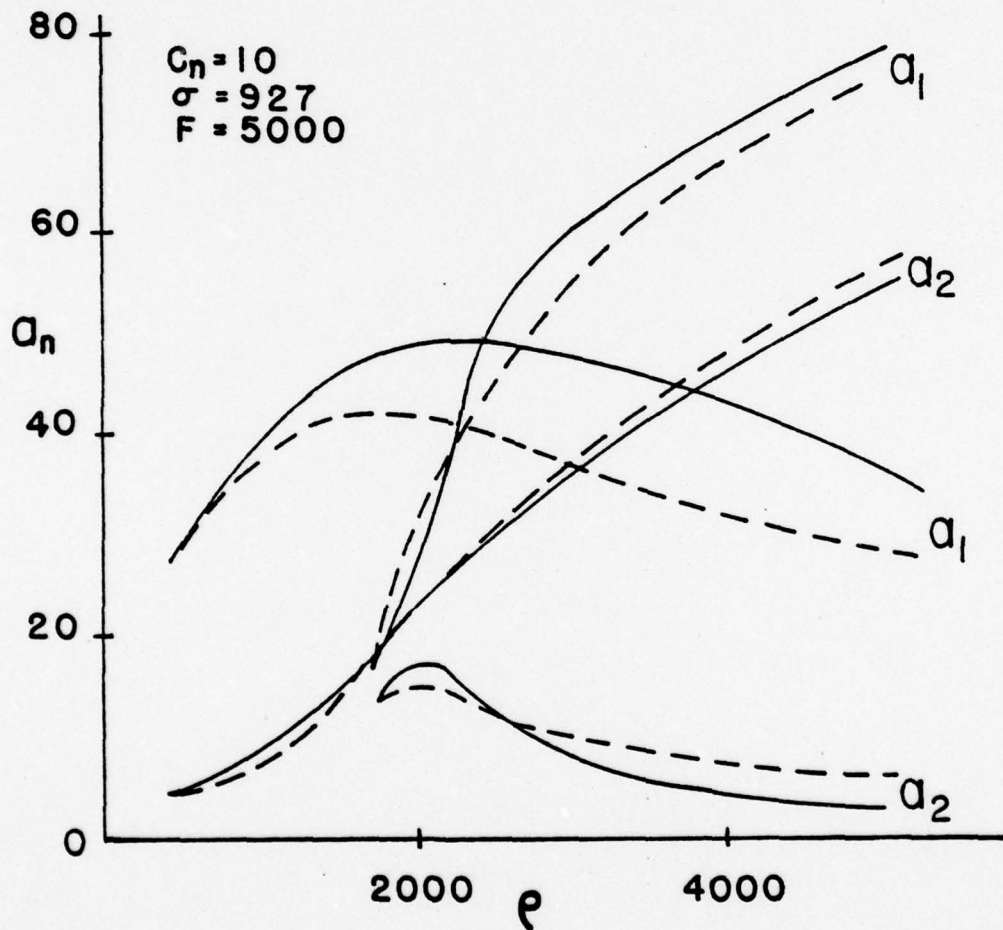


Figure 9. The modal amplitudes as functions of the frequency of the excitation,  $\rho(\varepsilon\rho = \lambda - \omega_1 - \omega_2)$ , for constant amplitude of the excitation. — stable, ---- unstable.



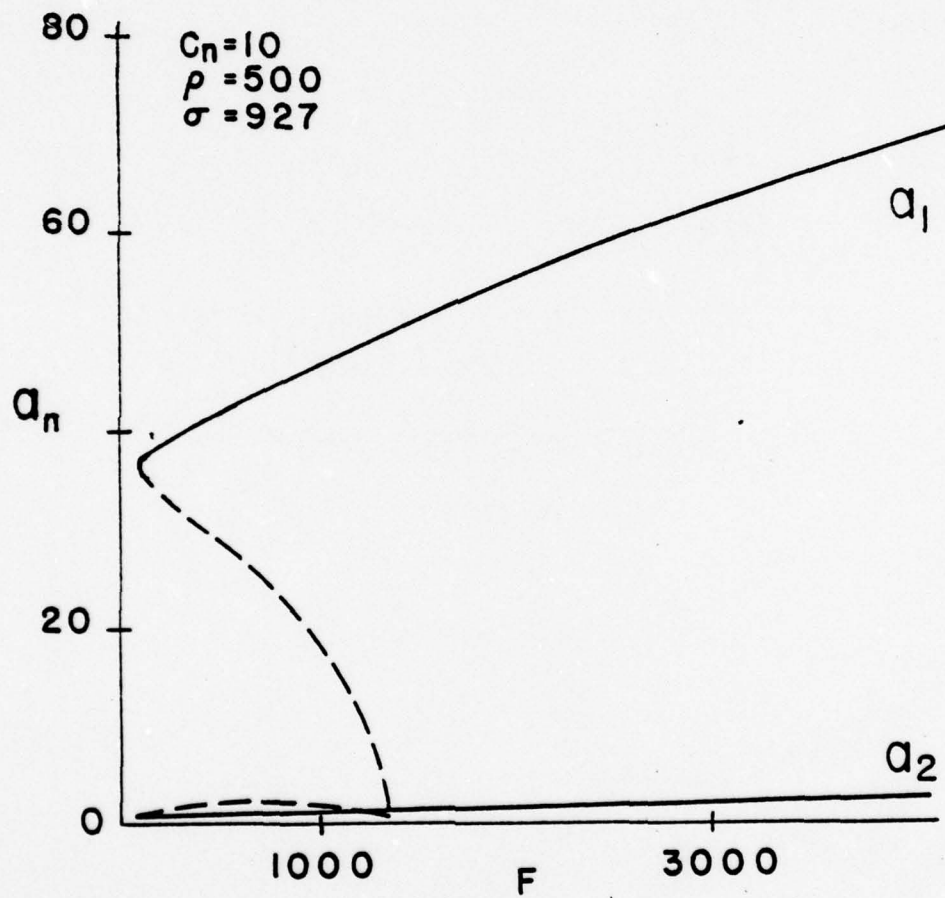


Figure 10. The modal amplitudes as functions of the amplitude of the excitation,  $F$ , for constant frequency of the excitation,  $\rho(\varepsilon\rho = \lambda - 2\omega_1)$ .  
 — stable, ---- unstable.

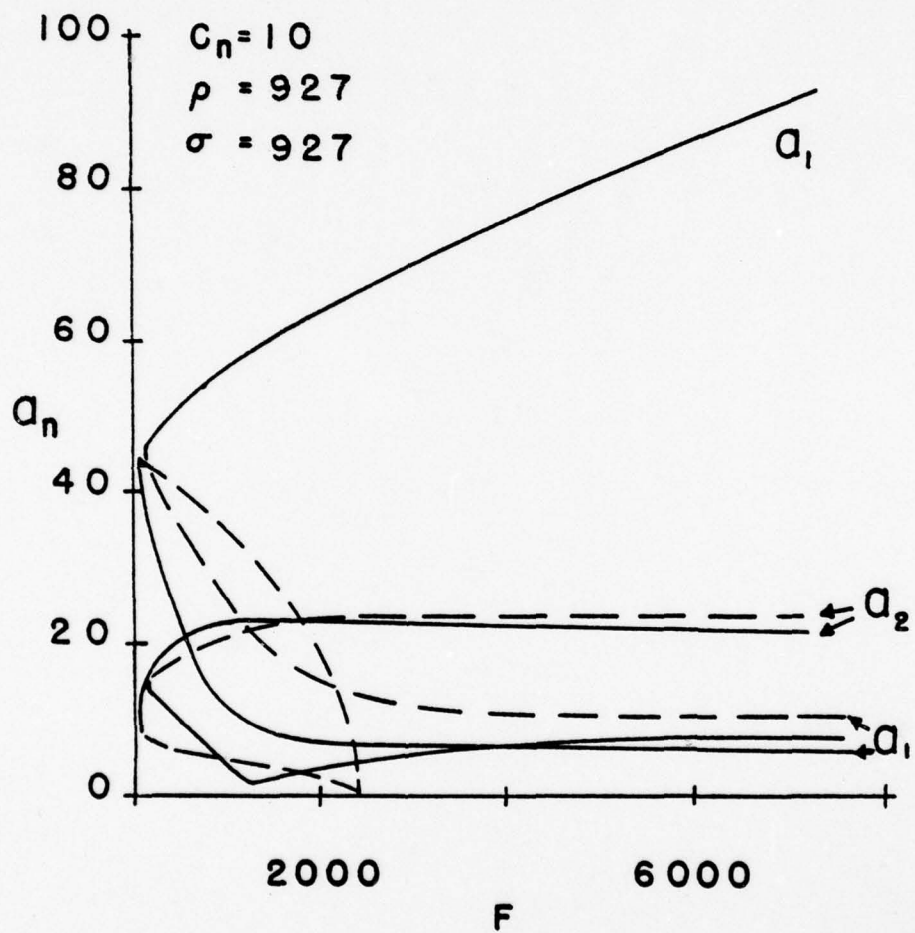


Figure 11. The modal amplitudes as functions of the amplitude of the excitation,  $F$ , for constant frequency of the excitation,  $\rho(\epsilon\rho = \lambda - 2\omega_1)$ . — stable, ---- unstable.

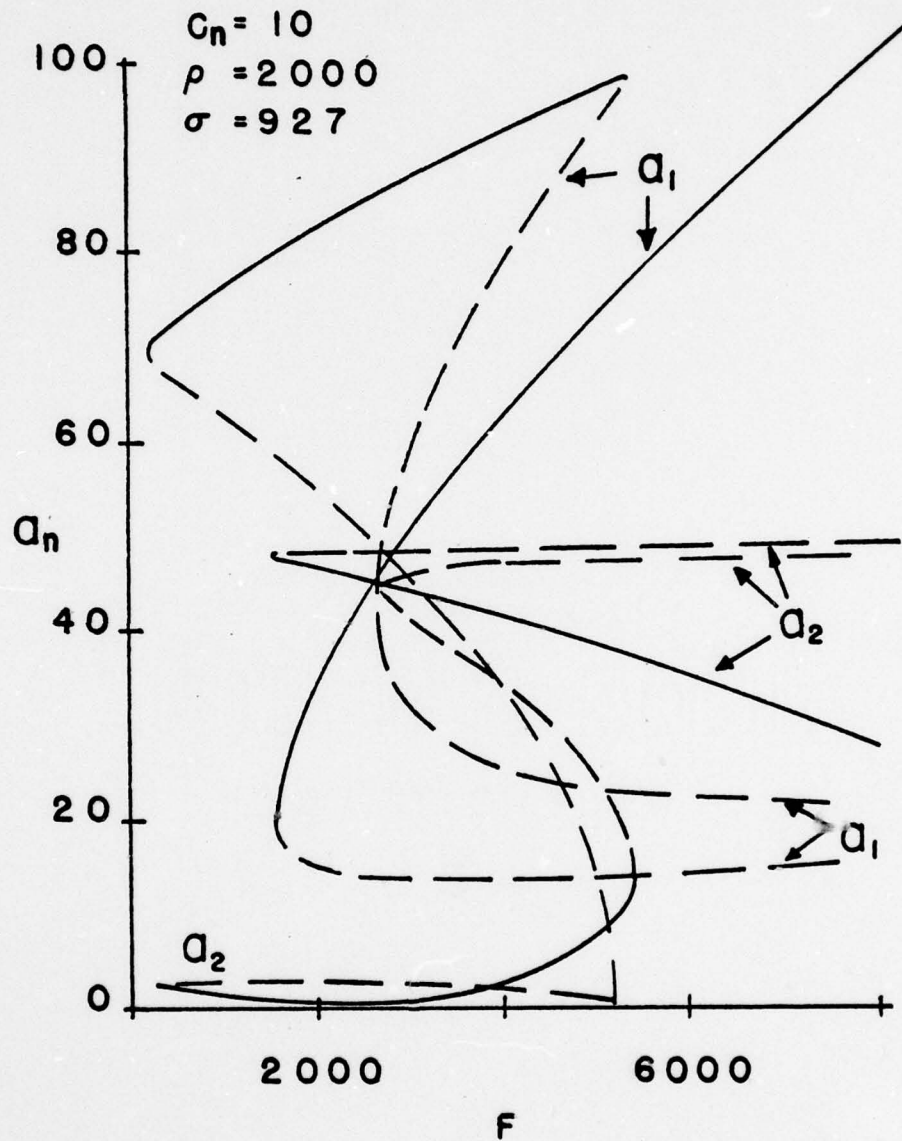


Figure 12. The modal amplitudes as functions of the amplitude of the excitation,  $F$ , for constant frequency of the excitation,  $\rho(\varepsilon\rho = \lambda - 2\omega_1)$ .  
 — stable, ---- unstable.



first mode, over a very limited range of the amplitude of the excitation. However, strong modal coupling is present, and the second mode amplitude becomes relatively significant at the expense of the first; hence we note energy being transferred from the excited mode to the one involved in the internal resonance. Figures 4, 10, 11, and 12 graphically demonstrate the transition from a single-mode response to a multimode response. The larger the detuning  $\rho$  is, the more the second mode enters the response.

b. The Effect of Damping and the Amplitudes of the Excitation on the Response,  $a_n$  vs  $\rho$ .

Using Figure 5 as a point of departure, we see that the effects of damping on the system can be made apparent. As a general rule, the smaller the damping coefficients  $C_n$  are, the more interaction one can expect. Increasing the damping coefficients tends to reduce the modal interaction to the point where the second-mode response is below that of the first mode, and there is no longer a crossover point between  $a_1$  and  $a_2$ . In addition, jump phenomena tend to disappear as damping increases. Figures 13 and 14 demonstrate both of these effects. As the damping increases, the response approaches that of Figure 2 as a limit, or the response approaches the single-mode response without internal resonance. Increasing the amplitude of the excitation tends to spread the stable and unstable branches and straighten the curves, first the unstable branch, then the stable one. We note this behavior in comparing Figure 13 with Figure 14 and Figure 5 with Figure 15. As  $F$  increases beyond the zone where jumps can occur, (the central portion

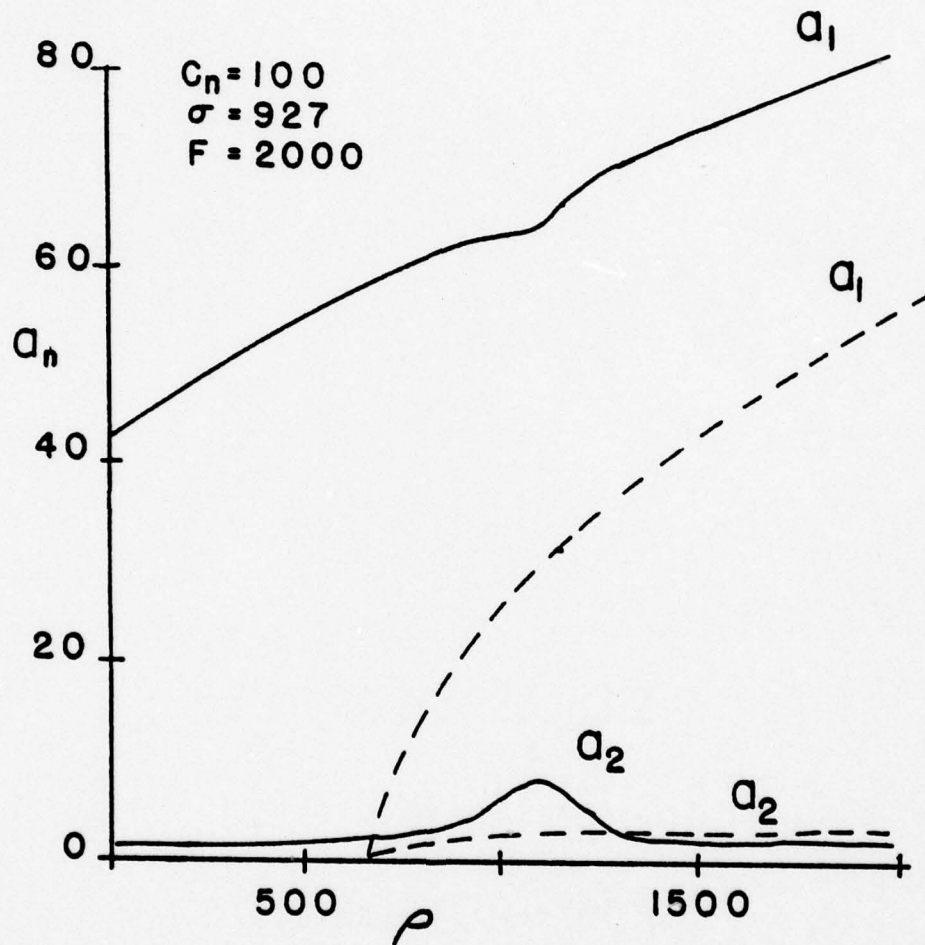


Figure 13. The modal amplitudes as functions of the frequency of the excitation,  $\rho(\epsilon\rho = \lambda - 2\omega_1)$ , for constant amplitude of the excitation,  $F$ . — stable, ---- unstable.

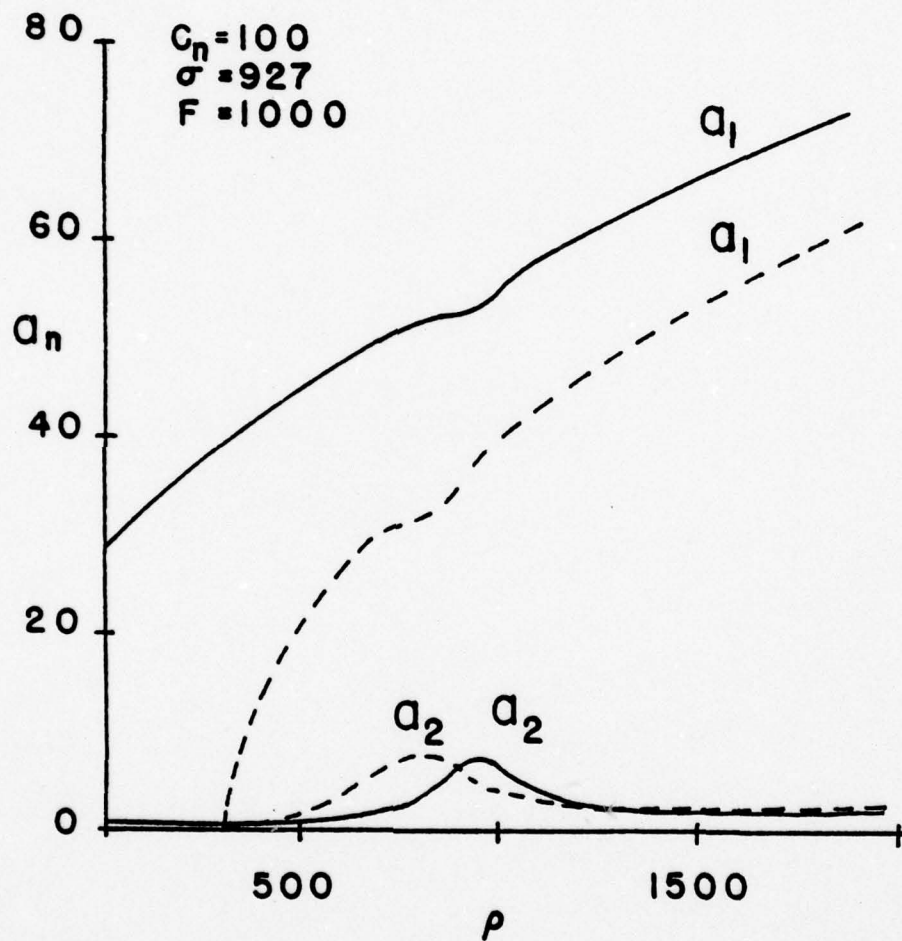


Figure 14. The modal amplitudes as functions of the frequency of the excitation,  $\rho(\epsilon\rho = \lambda - 2\omega_1)$ , for constant amplitude of the excitation,  $F$ . — stable, ---- unstable.



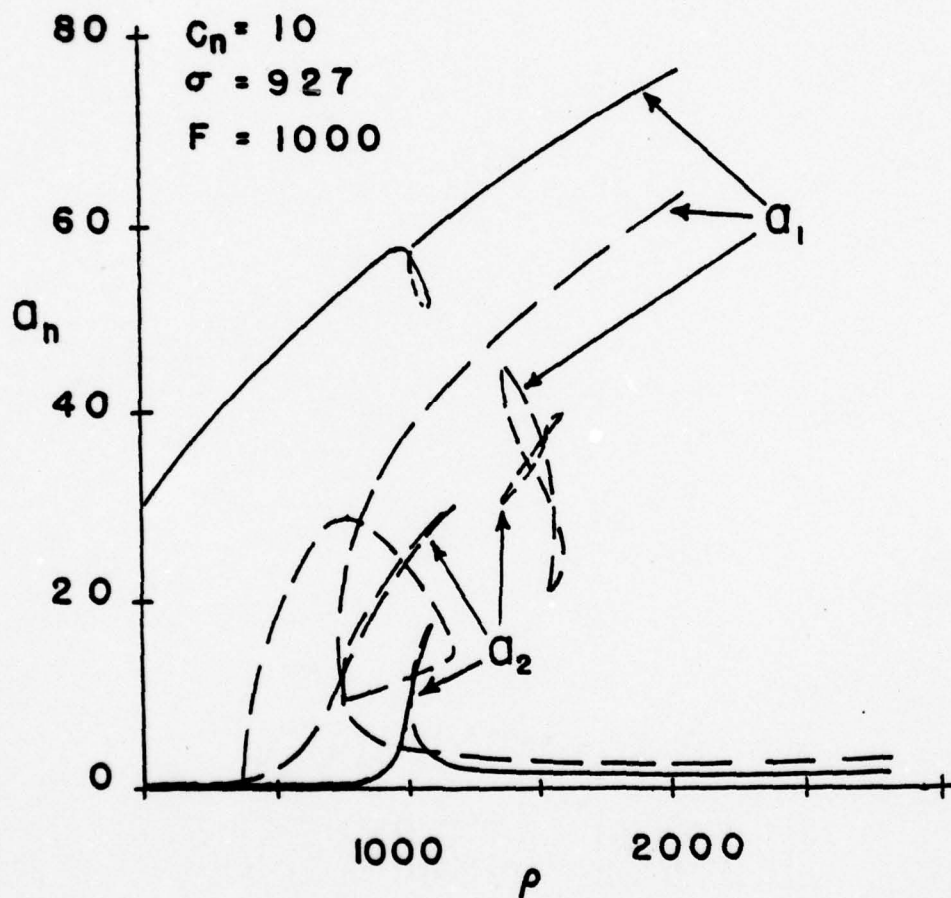


Figure 15. The modal amplitudes as functions of the frequency of the excitation,  $\rho(\epsilon\rho = \lambda - 2\omega_1)$ , for constant amplitude of the excitation,  $F$ . — stable, ---- unstable.

of Figure 4), then a further increase in  $F$  tends to force a single-mode response, approaching that of Figure 2.

### 8.5 Summary

The method of solution described in Chapters 2, 3, and 4 is used to solve two numerical examples -- one without internal resonances (a hinged-hinged column) and one with internal resonances (a hinged-clamped column).

For hinged-hinged columns, there are in effect no internal resonances and no combination resonances. Thus, the parametric excitation can, in the first approximation, excite only one mode. Jump phenomena can be produced by varying either the frequency or the amplitude of the excitation. In this case the frequency of the response is always one-half that of the excitation.

For hinged-clamped columns, there are both internal resonances and combination resonances. When  $\lambda$  is near  $2\omega_1$  or  $\omega_1 + \omega_2$ , both the first and second modes are strongly excited (i.e., appear in the first approximation), but generally the first mode dominates the second. Qualitatively, the response in both cases closely resembles a single-mode (the first) response. However, in the first case ( $\lambda$  near  $2\omega_1$ ) the frequency of the response is one-half that of the excitation, while in the second case it is one-fourth the frequency of the excitation.

When  $\lambda$  is near  $2\omega_2$ , two nontrivial solutions are possible -- either the second mode only is strongly excited or the second and first modes are strongly excited. In the second case, the first mode dominates, and the response qualitatively resembles the response when  $\lambda$  is

near  $2\omega_1$  and when  $\lambda$  is near  $\omega_1 + \omega_2$ . Multiple jumps are possible. When only the second mode is excited, the frequency of the response is one-half that of the excitation; on the other hand, when both modes are strongly excited, the frequency of the response is one-sixth that of the excitation.

Thus, for the cases considered, it is possible to produce responses having frequencies that are one-half, one-fourth, and one-sixth that of the excitation.

One excitation can produce rather dissimilar responses. On the other hand, several different excitations can produce responses which appear to be quite similar.

When  $\lambda$  is near  $2\omega_1$ , detuning the excitation allows the internal resonance to induce strong modal coupling and the second mode can be made to dominate the response. Increasing the damping and/or the amplitude of the excitation inhibits modal coupling. If the increase is large enough, it can result in essentially a single-mode response.



## CHAPTER NINE

### PANEL FLUTTER

The analysis contained in Chapters 5, 6, and 7. is applied to the motion of a plate in an airstream undergoing cylindrical bending (no spanwise bending). As in the last chapter, the governing equations are nondimensionalized, but here they are converted to the form discussed in Chapters 5, 6, and 7. The process is slightly more involved than for the beam.

#### 9.1 General

Following Dowell (1966, 1975), we write the governing equation as follows

$$D \frac{\partial^4 w}{\partial x^4} + \rho_m h \frac{\partial^2 w}{\partial t^2} = [N_x + \frac{Eh}{2a} \int_0^a (\frac{\partial w}{\partial \xi})^2 d\xi] \frac{\partial^2 w}{\partial x^2} - c \frac{\partial w}{\partial t} - \frac{\rho_\infty U_\infty^2}{(M^2 - 1)^{1/2}} \left[ \frac{\partial w}{\partial x} + \left( \frac{M^2 - 2}{M^2 - 1} \right) \frac{1}{U_\infty} \frac{\partial w}{\partial t} \right] \quad (9.1)$$

where  $D$  is the flexural rigidity,  $\rho_m$  is the material density,  $h$  is the panel thickness,  $a$  is the panel length,  $\rho_\infty$  is the density of the air in the freestream,  $U_\infty$  is the freestream velocity, and  $M$  is the Mach number, and  $N_x$  refers to the inplane parametric loading and is harmonic in what follows. The edges of the plate are restrained so that stretching occurs, and the term which accounts for it has the same cubic form as that of a beam. Modal damping is again assumed. The final term accounts for the aerodynamic pressure due to the plate motion and it is in the form of the so-called "piston-theory" approximation.

It is convenient to introduce nondimensional variables (denoted by primes); thus we let

$$\begin{aligned} x &= \frac{1}{2} a x', & w &= \frac{h^2}{a} w', & t &= \frac{a^2}{4} \frac{\rho_m h}{D} t' \\ \varepsilon^{\delta_F} R_x &= \frac{N_x a^2}{4D}, & \varepsilon^{\delta_N} &= \frac{3(1-\nu^2)h^2}{a^2}, & 2\varepsilon^{\delta_C} C' &= \frac{a^2}{4\sqrt{\rho_\infty h D}} C \end{aligned} \quad (9.2)$$

The parameter which is a measure of the dynamic pressure is

$$\tilde{\lambda} = \frac{\rho_\infty U_\infty^2 a^3}{8(M^2-1)^{1/2} D} \quad (9.3)$$

The coefficient of the last aerodynamic term is

$$\frac{(M^2-2)}{(M^2-1)^{5/4}} \frac{2\tilde{\lambda}\rho_\infty a}{\rho_m h} \quad (9.4)$$

and can be taken as small because the ratio of the density of air to the density of the material is small.

Substituting equations (9.2) and (9.3) into equation (9.1) and dropping the primes, we obtain

$$\begin{aligned} \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} + \tilde{\lambda} \frac{\partial w}{\partial t} &= [\varepsilon^{\delta_F} R_x + \varepsilon^{\delta_N} \int_0^2 (\frac{\partial w(\xi, t)}{\partial \xi})^2 d\xi] \frac{\partial^2 w}{\partial x^2} \\ &- 2\varepsilon^{\delta_C} C \frac{\partial w}{\partial t} \end{aligned} \quad (9.5)$$

where  $\delta_0$ ,  $\delta_F$ ,  $\delta_N$ , etc. conform with the ordering of Chapters 5-7.

We express the deflection as an expansion in terms of the linear free-oscillation modes; that is

$$w(x, t) = \sum_{m=1}^{\infty} U_m(t; \varepsilon) \phi_m(x) \quad (9.6)$$

The  $\phi_m$  are the orthonormal solutions of

$$\phi_m^{iv} - \omega_m^2 \phi_m = 0$$

where

$$\phi_m = 0 \quad \text{and} \quad \phi_m' = 0 \quad (9.7a)$$

at clamped edges and

$$\phi_m = 0 \quad \text{and} \quad \phi_m'' = 0 \quad (9.7b)$$

at hinged edges. The  $\omega_m$  are the natural frequencies and the  $\phi_m$  are the linear free-oscillation modes.

Substituting equation (9.6) into equation (9.5), multiplying by  $\phi_n$ , and integrating the result from  $x = 0$  to  $x = 2$ , we obtain

$$\begin{aligned} \ddot{U}_n + \omega_n^2 U_n + \tilde{\lambda} \sum_{m=1}^{\infty} \alpha_{nm} U_m = & [-\epsilon^{\delta F} \sum_{m=1}^{\infty} 2 \cos \lambda t f_{nm} U_m \\ & - 2\epsilon^{\delta C} C_n \dot{U}_n + \epsilon^{\delta N} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \Gamma_{nmpq} U_m U_p U_q] \end{aligned} \quad (9.8a)$$

where

$$\alpha_{nm} = \int_0^2 \phi_n \phi_m' dx \quad (9.8b)$$

$$R_x = F \cos \lambda t \quad (9.8c)$$

$$2f_{nm} = -F \int_0^2 \phi_n \phi_m'' dx = F \int_0^2 \phi_n' \phi_m' dx = FG_{nm} \quad (9.8d)$$

$$\Gamma_{nmpq} = \int_0^2 \phi_n \phi_m'' dx \int_0^2 \phi_p' \phi_q' dx = -G_{nm} G_{pq} \quad (9.8e)$$

At this stage with the beam, we had achieved the desired form of the governing equations, but this is not so for the flutter problem.



Equation (9.8a) may be written in matrix form as

$$\ddot{A}\ddot{U} + BU = \epsilon^{\delta_F}FU + \epsilon^{\delta_C}CU + \epsilon^{\delta_N}TU \quad (9.9)$$

provided we use a finite number of terms in the summation. The matrices  $\ddot{U}$  and  $U$  are column matrices of length  $M$ . The remainder are square matrices of dimension  $M \times M$ . Then we note that

$$[A] = [I] \quad (9.10a)$$

The identity matrix

$$[B] = [\omega_n^2] + \tilde{\lambda}[\alpha_{nm}] \quad (9.10b)$$

where the frequency matrix is diagonal,

$$[F] = -2\cos\lambda t[f_{nm}] \quad (9.10c)$$

$$[C] = -2[C_{nm}] \quad (9.10d)$$

and

$$[T] = -[U]^T [G_{nm}][U][G_{pq}] \quad (9.10e)$$

The system of equations which has been analyzed in Chapters 5 through 7 is of the form

$$I\ddot{U} + JU = \epsilon^{\delta_F}F'U + \epsilon^{\delta_C}C'U + \epsilon^{\delta_\Lambda} \Lambda B'U + \epsilon^{\delta_N}T'U \quad (9.11a)$$

where the first coefficient matrix is the identity matrix and

$$[J] = \begin{bmatrix} \omega_1^2 & 0 & 0 & 0 & . & . & . & 0 \\ 1 & \omega_1^2 & 0 & . & . & . & . & 0 \\ 0 & 0 & \omega_2^2 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & \omega_M^2 \end{bmatrix} \quad (9.11b)$$

is of the Jordan canonical form, which, in this case is a diagonal matrix with one off-diagonal term that is non-zero;  $F'$ ,  $C'$  and  $T'$  contain the same type of terms as their counter parts in equation (9.9).

The matrix  $B'$  contains aerodynamic coefficients which appear when the dynamic pressure causes the stability parameter  $\tilde{\lambda}$  to be near the critical value  $\tilde{\lambda}_c$  which causes two natural frequencies to merge and flutter to ensue. A refinement of the matrix  $B'$  is required before an attempt can be made to transform equation (9.9) into equation (9.11a), the Jordan form. Equation (9.10b) may be written

$$[B] = [\omega_n^2] + \tilde{\lambda}_c [\alpha_{nm}] - (\tilde{\lambda}_c - \tilde{\lambda}) [\alpha_{nm}] \quad (9.12a)$$

$$= [B_c] + (\tilde{\lambda} - \tilde{\lambda}_c) [\alpha_{nm}] \quad (9.12b)$$

where  $\tilde{\lambda} - \tilde{\lambda}_c$  is a measure of the nearness of  $\tilde{\lambda}$  to the critical value  $\tilde{\lambda}_c$ , and can be expressed in terms of a detuning parameter as follows

$$\frac{\delta \Lambda}{\epsilon} = \tilde{\lambda} - \tilde{\lambda}_c \quad (9.12c)$$

Then it is actually  $B_c$  which must be converted to Jordan form when the value of the stability parameter reaches the critical value,  $\tilde{\lambda}_c$ , and the onset of flutter occurs. The two lowest eigenfrequencies of  $B_c$  become equal at this time. The larger eigenfrequencies remain very near the  $\omega_n$  for  $n \geq 3$ .

It is now possible to introduce a similarity transformation which transforms the matrix  $B_c$  to a Jordan form. (A complete discussion of the choice of this matrix is contained in Appendix A). We thus choose a matrix  $Z$  such that

$$U = ZV \quad (9.13a)$$

Then the similarity transformation that leaves the eigenvalues unchanged becomes

$$Z^{-1}B_C ZV = JV = \begin{bmatrix} \omega_2 & 0 & 0 & 0 & . & . & 0 \\ 1 & \omega^2 & 0 & 0 & . & . & 0 \\ 0 & 0 & \omega_3^2 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & \omega_M^2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ . \\ . \\ . \\ V_M \end{bmatrix} \quad (9.13b)$$

$$Z^{-1}AZ\ddot{V} = Z^{-1}I\ddot{V} = I\ddot{V} \quad (9.13c)$$

$$Z^{-1}FZV = F'V \quad (9.13d)$$

and

$$Z^{-1}TZV = T'V \quad (9.13e)$$

This last transformation merits some detailed consideration. From equation (9.13a) we can write the transpose as

$$U^T = V^T Z^T \quad (9.13f)$$

Then using equations (9.10e) and (9.13e) we write

$$Z^{-1}TZV = Z^{-1}V^T Z^T G_{nm} ZV G_{pq} ZV \quad (9.13g)$$

Letting

$$G_1 = Z^T G_{nm} Z \quad (9.13h)$$

we note that

$$V^T G_1 V \quad (9.13i)$$

is a quadratic form and it is thus a scalar. Hence we may write equation (9.13g) as

$$V^T G_1 V Z^{-1} G_{pq} ZV \quad (9.13j)$$



Letting

$$G_2 = Z^{-1} G_{pq} Z \quad (9.13k)$$

we write

$$T'V = -V^T G_1 V G_2 V \quad (9.13l)$$

Equation (9.13l) has the same cubic form as before, or

$$\sum_m \sum_p \sum_q \Gamma'_{nmpq} V_m V_p V_q \quad (9.14a)$$

where

$$\Gamma'_{nmpq} = G_{1nm} G_{2pq} \quad (9.14b)$$

Finally, the second half of the aero-matrix of equation (9.12b) can be transformed into

$$B' = -Z^{-1}[\alpha]Z = \beta \quad (9.15)$$

Hence

$$\ddot{IV} + JV = \epsilon_F^{\delta} F'V + \epsilon_C^{\delta} C'V + \epsilon_T^{\delta} T'V + \epsilon_{\Lambda\beta V}^{\delta} \quad (9.16)$$

and the panel flutter problem has been reduced to the form of equations (5.1). We shall now discuss the response for the various cases of parametric and internal resonances examined in Chapters 5-7.

## 9.2 Simply Supported Plate.

For cylindrical bending of a simply supported plate we may write

$$\phi_n = \sin \sqrt{\omega_n} x \quad (9.17a)$$

and

$$\omega_n = \frac{1}{4} n^2 \pi^2 \quad (9.17b)$$

where the  $\phi_n$  and  $\omega_n$  are the free-oscillation modes and frequencies. It follows from equation (9.8) that

$$\alpha_{nm} = \begin{cases} \frac{2nm}{n^2 - m^2} & \text{if } m + n \text{ is odd} \\ 0 & \text{if } m + n \text{ is even} \end{cases} \quad (9.18a)$$

$$f_{nm} = \frac{1}{2} \omega_n F \delta_{nm} \quad (9.18b)$$

$$G_{nm} = \frac{1}{2} \omega_n \delta_{nm} \quad (9.18c)$$

$$\Gamma_{nmpq} = -G_{nm} G_{pq} \quad (9.18d)$$

The transformations of equations (9.13) are applied to equations (9.18) to provide

$$\beta_{12} = 99.2 \quad (9.19a)$$

$$f_{12} = 404.9 \frac{F}{2} \quad (9.19b)$$

$$f_{13} = -3917 \frac{F}{2} \quad (9.19c)$$

$$f_{32} = .1017 \frac{F}{2} \quad (9.19d)$$

$$f_{33} = 22.74 \frac{F}{2} \quad (9.19e)$$

$$\Gamma_{1222} = -5729 \quad (9.19f)$$

and

$$\Gamma_{3222} = -1.438 \quad (9.19g)$$

The frequencies become

$$\omega_1 = \omega_2 = 8.1076 \quad (9.20a)$$

$$\omega_3 = 22.066 \quad (9.20b)$$

While the critical value of the dynamic-pressure parameter is

$$\tilde{\lambda}_c = 42.917 \quad (9.20c)$$

where the  $\omega_n$  and  $\tilde{\lambda}_c$  satisfy

$$\phi^{iv} + \tilde{\lambda}_c \phi' - \omega_n^2 \phi = 0 \quad (9.21)$$

The various cases of Chapters 5-7 are discussed, beginning with the response in the presence of a parametric resonance. The cases considered are

(1)  $\lambda$  near  $2\omega_1$

(2)  $\lambda$  near  $\omega_3 \pm \omega_1$

### 9.3 Parametric Resonance

#### a. Case (1) $\lambda \approx 2\omega_1$

The governing equations for this case were developed in Section 6.1, and for convenience we rewrite equations (6.11) as

$$a_2^2 = \left[ \frac{4}{3\Gamma_{jkkk}} (-\Lambda B_{12} + \omega_1^2 (4C_1 C_2 - \rho^2) \pm \sqrt{f_{12}^2 - 4\omega_1^4 \rho^2 (C_1 + C_2)^2}) \right] \quad (9.22a)$$

$$a_1 = \omega_1 a_2 \sqrt{\rho^2 + 4C_2^2} \quad (9.22b)$$

A general observation that can be made immediately concerns the relative sizes of  $a_1$  and  $a_2$ . If

$$\sqrt{\rho^2 + 4C_2^2} < \frac{1}{\omega_1} \quad (9.23a)$$

Hence, for  $a_2 > a_1$  the detuning must be very nearly zero and the damping extremely small, otherwise  $a_1 > a_2$ . We note that even though  $a_1 > a_2$



the response is not necessarily dominated by the first mode. The modal amplitude  $a_1$  multiplies a one (1) whereas  $a_2$  multiplies  $\varepsilon^{-\delta_0}$  and hence its effect can be much greater. Figure 16 shows the variation of the modal amplitude with the amplitude of excitation and the response has essentially the same form as that in Figure 3. The response contains the same regions where trivial and nontrivial solutions exist and jump phenomena are possible. We note that as the amplitude of the parametric resonance term increases,  $0 \leq F \leq F_1$ , the only solution is the trivial one and it is stable. Then we enter the region where two nontrivial solutions exist,  $F_1 \leq F \leq F_2$ , one stable and one unstable. Finally when  $F \geq F_2$  the trivial solution is unstable and only one nontrivial solution exists, which is stable, and a jump occurs. If we scan in the opposite direction we start with a nontrivial stable solution and remain on that upper branch until we reach the region where the only solution is the stable trivial solution and a jump occurs. One can refer to Figure 3 where the arrows indicate the jump phenomena and the regions where trivial and nontrivial solutions exist are shown. The form of the response for the second mode is the same as the first except the amplitude is so small that it appears to be zero by comparison.

The analog of Figure 2, modal amplitude vs detuning of the excitation frequency, is Figure 17. This figure shows the modal amplitude  $a_1$  vs the detuning of the dynamic-pressure parameter,  $\Lambda$ . However, the form of the response is essentially the same and exhibits the same regions of stability and jump phenomena. As we scan from large negative  $\Lambda$  to large positive  $\Lambda$ , or alternatively, as we increase the dynamic pressure from below the flutter speed, we encounter first the region

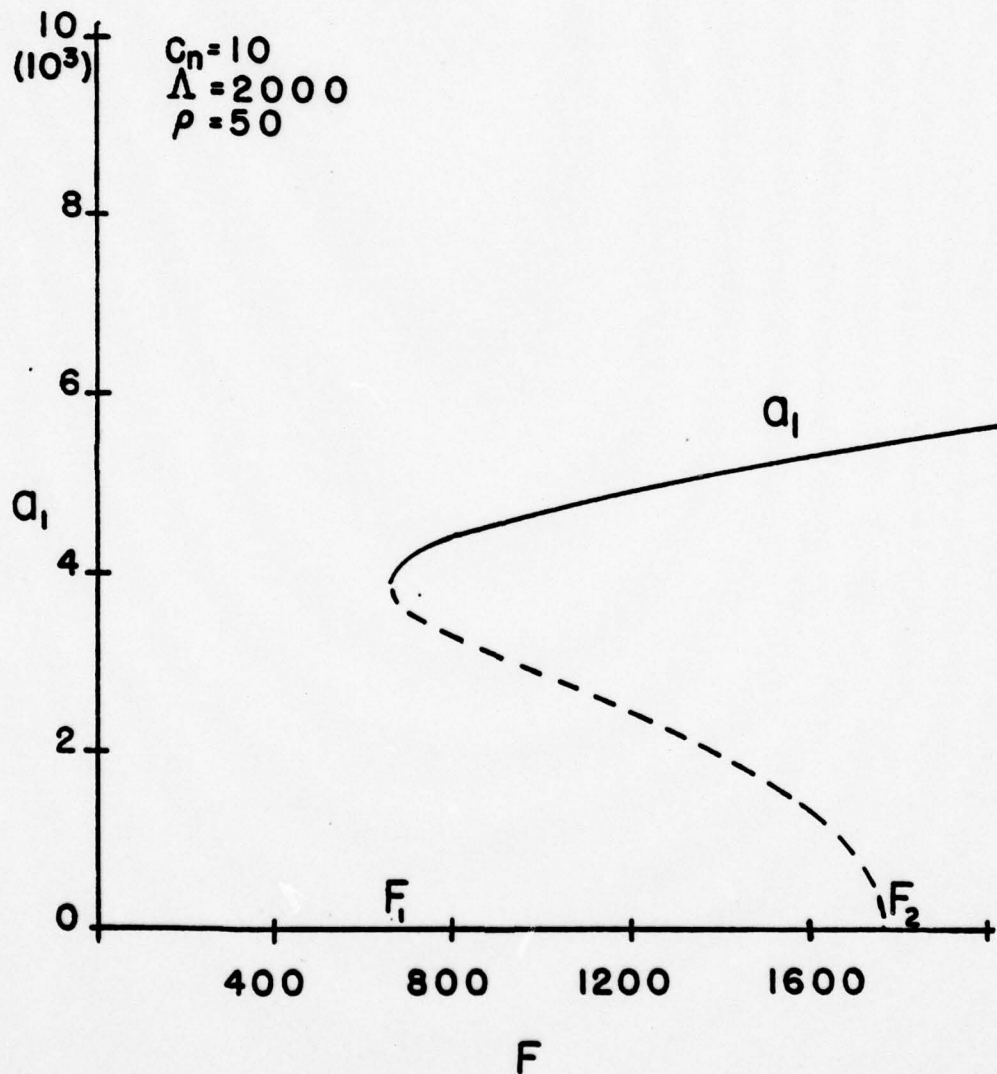


Figure 16. First modal amplitude as a function of the amplitude of the excitation,  $F$ , for constant frequency of the excitation  $\rho(\epsilon\rho = \lambda - 2\omega_1)$ , and constant aerodynamic detuning,  $\Lambda$ .  
 — stable, ---- unstable.

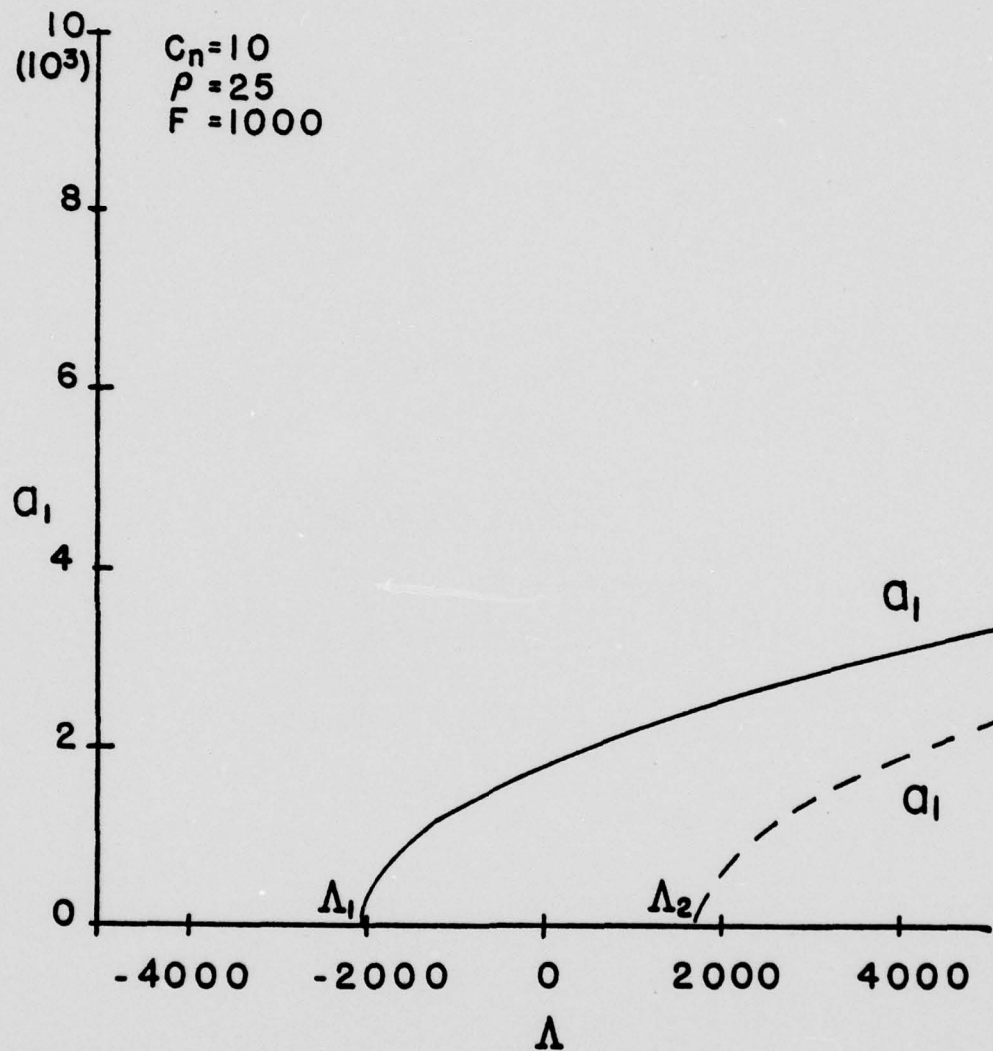


Figure 17. First modal amplitude as a function of aerodynamic detuning,  $\Lambda$ , for constant frequency of excitation  $\rho(\epsilon\rho = \lambda - 2\omega_1)$  and constant amplitude of excitation,  $F$ . — stable, ---- unstable.



where the trivial solution is the only solution and it is stable,  $\Lambda \leq \Lambda_1$ . Between  $\Lambda_1 \leq \Lambda \leq \Lambda_2$  the trivial solution is unstable and the only nontrivial solution is stable. Beyond  $\Lambda > \Lambda_2$  there is again a stable trivial solution and two nontrivial solutions, only the larger one being stable. Again the form of the second mode is essentially the same except the magnitude appears to be zero by comparison.

In Figure 18 we plot the modal amplitude vs the detuning of the parametric excitation  $\rho$ . This new graph has a character of its own and has no analog with the case of distinct frequencies. However, it still contains regions where multiple solutions exist and points out where jumps can occur. We can scan the frequency axis and observe the following. For  $\rho \leq \rho_1$  only the trivial solution exists and it is stable. Between  $\rho_1 \leq \rho \leq \rho_2$  two nontrivial solutions exist, the larger being stable, the smaller unstable. In addition, the trivial solution is stable, and since we are scanning from the left, the response remains trivial. Between  $\rho_2 \leq \rho \leq \rho_3$  only one nontrivial solution exists and it is stable. Concurrently, the trivial solution becomes unstable and an upward jump occurs. Arrows indicate the jumps and paths followed by  $a_1$ . As the frequency increases between  $\rho_3 \leq \rho \leq \rho_4$  the response remains nontrivial, since that solution is stable. A nontrivial, unstable solution also exists as well as a stable trivial solution. For  $\rho > \rho_4$  we see that the only solution possible is the trivial one, which is stable, and a downward jump occurs where the nontrivial solution ceases to exist. Scanning from the right we note a mirror-image response. The solution is first trivial until  $\rho = \rho_3$ . It jumps to the nontrivial

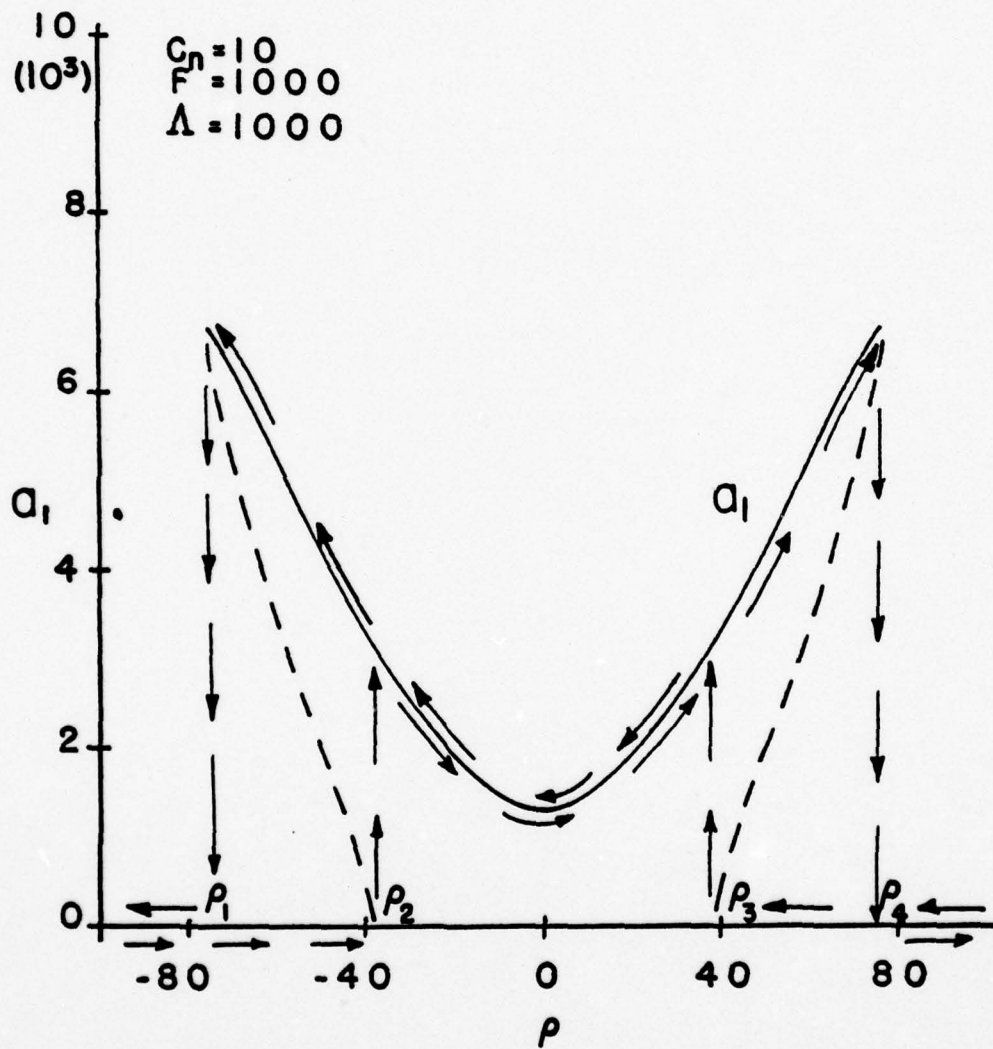


Figure 18. First modal amplitude as a function of frequency of excitation,  $\rho$  ( $\epsilon\rho = \lambda - 2\omega_1$ ), for constant amplitude of excitation,  $F$ , and constant aerodynamic detuning,  $\Lambda$ . — stable, ---- unstable.

solution and remains nontrivial until  $\rho = \rho_1$  at which time it jumps back to the trivial solution for further decreases in  $\rho$ .

When the damping is small and the frequency of excitation is almost exactly tuned, or  $\rho$  is very near zero, the condition of equation (9.23a) can be met. The second mode is greater than the first mode and we note from Figures 19 and 20 that the characteristic shape of the curves is the same. Hence, the amount of detuning and damping will determine which mode amplitude is larger.

It should be noted that for the plate considered,  $\Gamma_{1222}$  is negative in equations (9.22a) and the existence of any solution other than the trivial one depends upon the conditions

$$\Lambda \beta_{12} - \omega_1^2(4C_1C_2 - \rho^2) \pm \sqrt{f_{12}^2 - 4\omega_1\rho^2(C_1 + C_2)^2} > 0 \quad (9.24a)$$

and

$$f_{12}^2 > 4\omega_1\rho^2(C_1 + C_2)^2 \quad (9.24b)$$

If (9.24b) is not satisfied then there are no nontrivial solutions. The same holds for (9.24a). If both conditions are met then there are two nontrivial solutions. The stable one is associated with the positive value of the radical. When only one solution exists, it is also associated with the positive value of the radical. In addition, it is dependent on the magnitude of the amplitude,  $F$ , of the parametric excitation, the relative sizes of the detuning and the damping, and the aerodynamic detuning parameter,  $\Lambda$ . We note that in this case,  $\beta_{12}$  is a positive constant and that a positive value of  $\Lambda$  refers to a dynamic pressure in excess of the critical flutter speed. The presence of large damping tends to necessitate large forcing amplitudes, large detunings of the parametric terms, and large values of dynamic pressures to produce



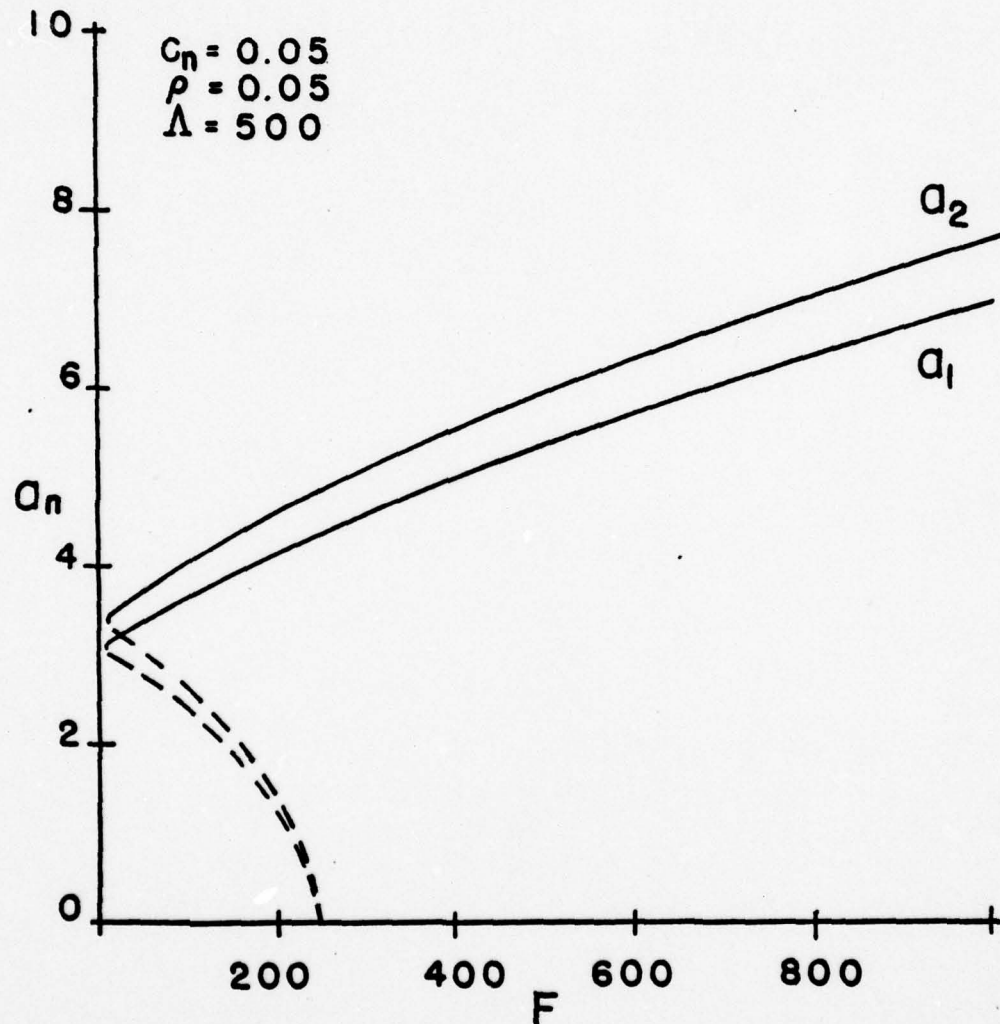


Figure 19. The modal amplitudes as functions of the amplitude of excitation,  $F$ , for constant frequency of excitation  $\rho(\epsilon\rho = \lambda - 2\omega_1)$  and constant aerodynamic detuning,  $\Lambda$ .  
 — stable, ---- unstable.

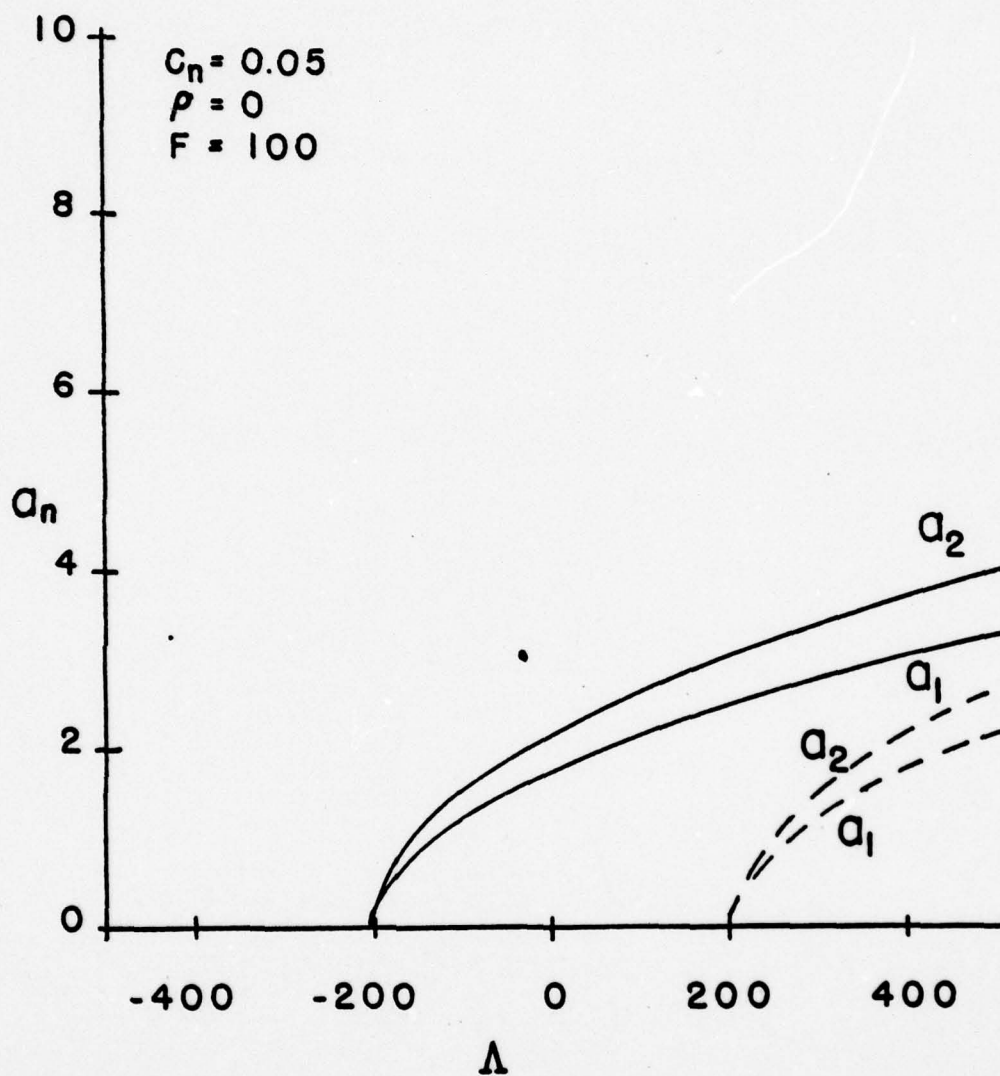


Figure 20. The modal amplitudes as functions of the aerodynamic detuning,  $\Lambda$ , for constant frequency of excitation,  $\rho(\epsilon\rho = \lambda - 2\omega_1)$  and constant amplitude of excitation,  $F$ . — stable, ---- unstable.

nontrivial solutions. However, this same condition make  $a_2$  very small compared with  $a_1$ . On the other hand, small damping permits strong interaction of the modes and even allows  $a_2$  to be much greater than  $a_1$  when detuning is small. The amplitude of the parametric excitation and the aerodynamic detuning merely influence the amplitude of the response but not the relative sizes of  $a_1$  and  $a_2$ .

A few special cases may also be examined. If no parametric resonance is present, (9.24a) has the form

$$\Lambda\beta_{12} - 4C_1C_2\omega_1^2 > 0 \quad (9.25)$$

and there is either the trivial solution or one nontrivial solution, depending on whether (9.25) is satisfied. Figure 21 illustrates the response for small damping. When large damping is present  $a_1 > a_2$ . As  $C_2 \rightarrow 0$ ,  $a_1 \rightarrow 0$  and the only mode excited is the second, in spite of the parametric force exciting the first mode. The response is as shown in Figure 21 without  $a_1$  being present.

B. Case (2)  $\lambda$  near  $\omega_3 \pm \omega_1$ .

The governing equations for these cases were developed in Sections (6.2) and (6.3) and are

$$\omega_1 C_1 a_1 + \frac{f_{13}}{2} a_3 \sin \beta_1 + \left[ \frac{3}{8} \Gamma_{1222} a_2^3 + \frac{\Lambda}{2} \beta_{12} a_2 \right] \sin \gamma = 0 \quad (9.26a)$$

$$\omega_1 C_2 a_2 + \frac{a_1}{2} \sin \gamma = 0 \quad (9.26b)$$

$$\omega_3 C_3 a_3 + \frac{f_{32}}{2} a_2 \sin \beta_2 = 0 \quad (9.26c)$$

$$\rho = \frac{a_1}{2\omega_1 a_1} \cos \gamma + \frac{f_{32} a_2}{2\omega_3 a_3} \cos \beta_2 \quad (9.26d)$$

$$a_1^2 \cos \gamma = 2a_2 \left\{ \frac{f_{13}}{2} a_3 \cos \beta_1 - \left[ \frac{3}{8} \Gamma_{1222} a_2^3 + \frac{\Lambda}{2} \beta_{12} a_2 \right] \cos \gamma \right\} \quad (9.26e)$$



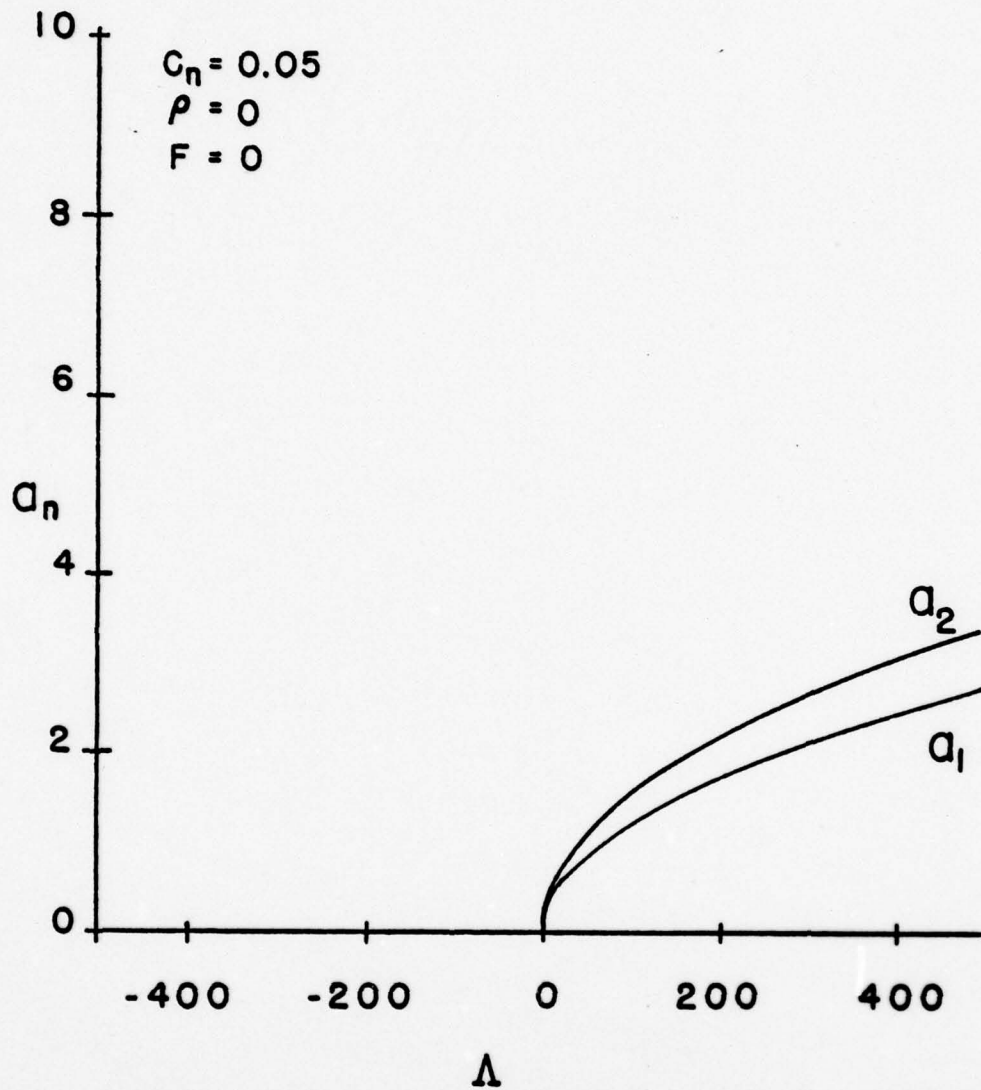


Figure 21. The modal amplitudes as functions of the aerodynamic detuning,  $\Lambda$ , for no parametric resonance.

— stable, ---- unstable.

where

$$\beta_1 = \begin{cases} \rho T_2 - \alpha_r - \alpha_j & \text{for } \omega_3 + \omega_1 \\ -\rho T_2 + \alpha_r - \alpha_j & \text{for } \omega_3 - \omega_1 \end{cases} \quad (9.27a)$$

$$\gamma = \alpha_j - \alpha_k \quad (9.27b)$$

$$\beta_2 = \begin{cases} \rho T_2 - \alpha_r - \alpha_k & \text{for } \omega_3 + \omega_1 \\ -\rho T_2 - \alpha_r + \alpha_k & \text{for } \omega_3 - \omega_1 \end{cases} \quad (9.27c)$$

Equations (9.26) do not readily admit a closed-form solution so a Newton-Raphson technique was used to produce Figures 22, 23, and 24 for the case of  $\lambda$  near  $\omega_1 + \omega_3$ . A similar response to that of the last section is evident. In the presence of large damping,  $a_1 > a_2$  and  $a_3$ . Regions of stability exist as well as jump phenomena. Upon examining the response, amplitude vs detuning, Figure 24, we note that there is a loss of symmetry from the case of  $\lambda$  near  $2\omega_1$ .

#### 9.4 Internal Resonance, $\omega_3 \approx 3\omega_1$

We now investigate the response in the presence of an internal resonance of the form,  $\omega_3$  near  $3\omega_1$ . For the simply supported case this is within about 10%. By a proper choice of the boundary conditions, for example some form of elastic supports, it would be possible to obtain a relationship within much less than 10%. We now investigate the response in the presence of an internal resonance for the case of parametric resonance:

- (1)  $\lambda$  near  $2\omega_1$
- (2) no parametric resonance

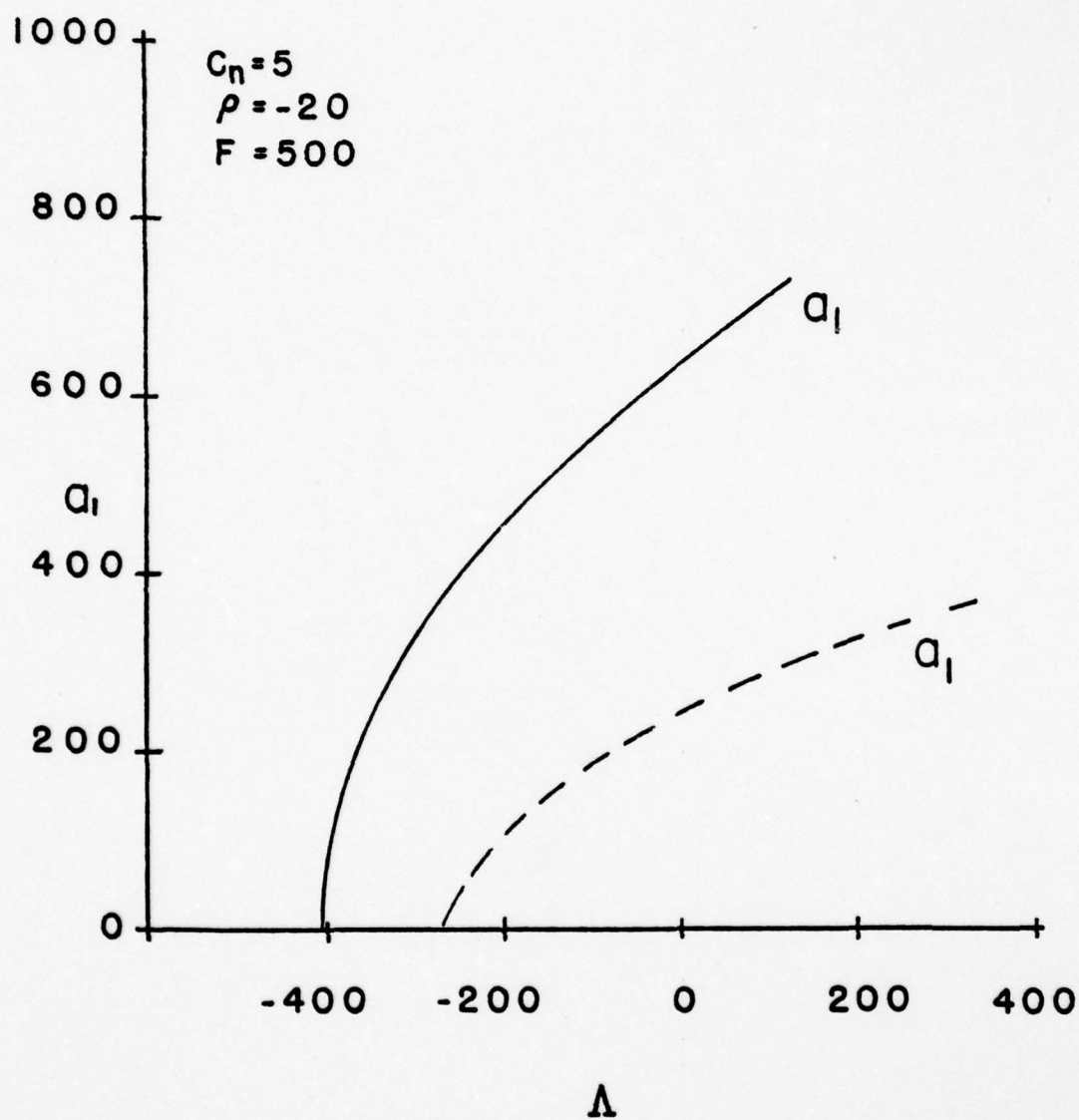


Figure 22. First modal amplitude as a function of aerodynamic detuning,  $\Lambda$ , for constant frequency of excitation,  $\rho(\epsilon\rho = \lambda - \omega_1 - \omega_3)$ , and constant amplitude of excitation,  $F$ . — stable, ---- unstable.



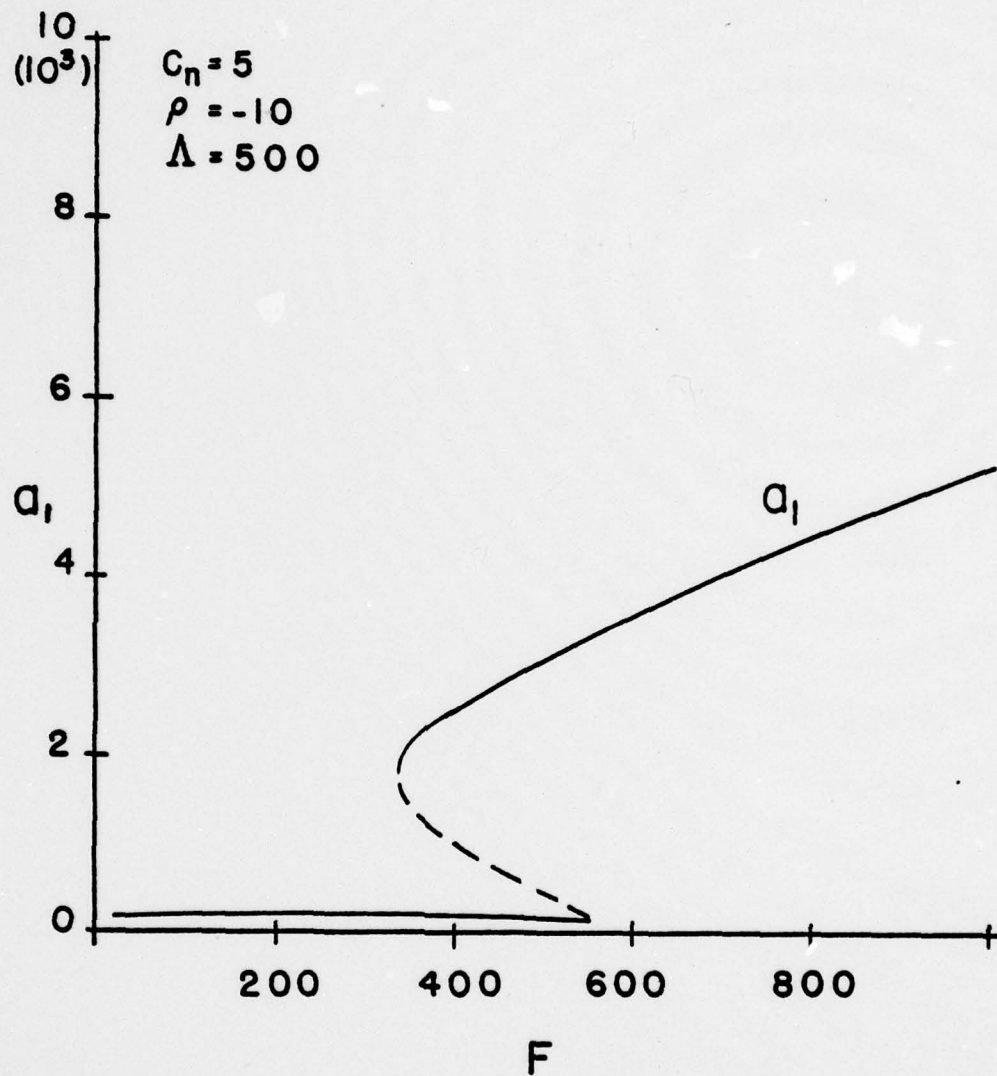


Figure 23. First modal amplitude as a function of amplitude of excitation,  $F$ , for constant frequency of excitation,  $\rho(\epsilon\rho = \lambda - \omega_1 - \omega_3)$  and constant aerodynamic detuning,  $\Lambda$ . — stable, ---- unstable.

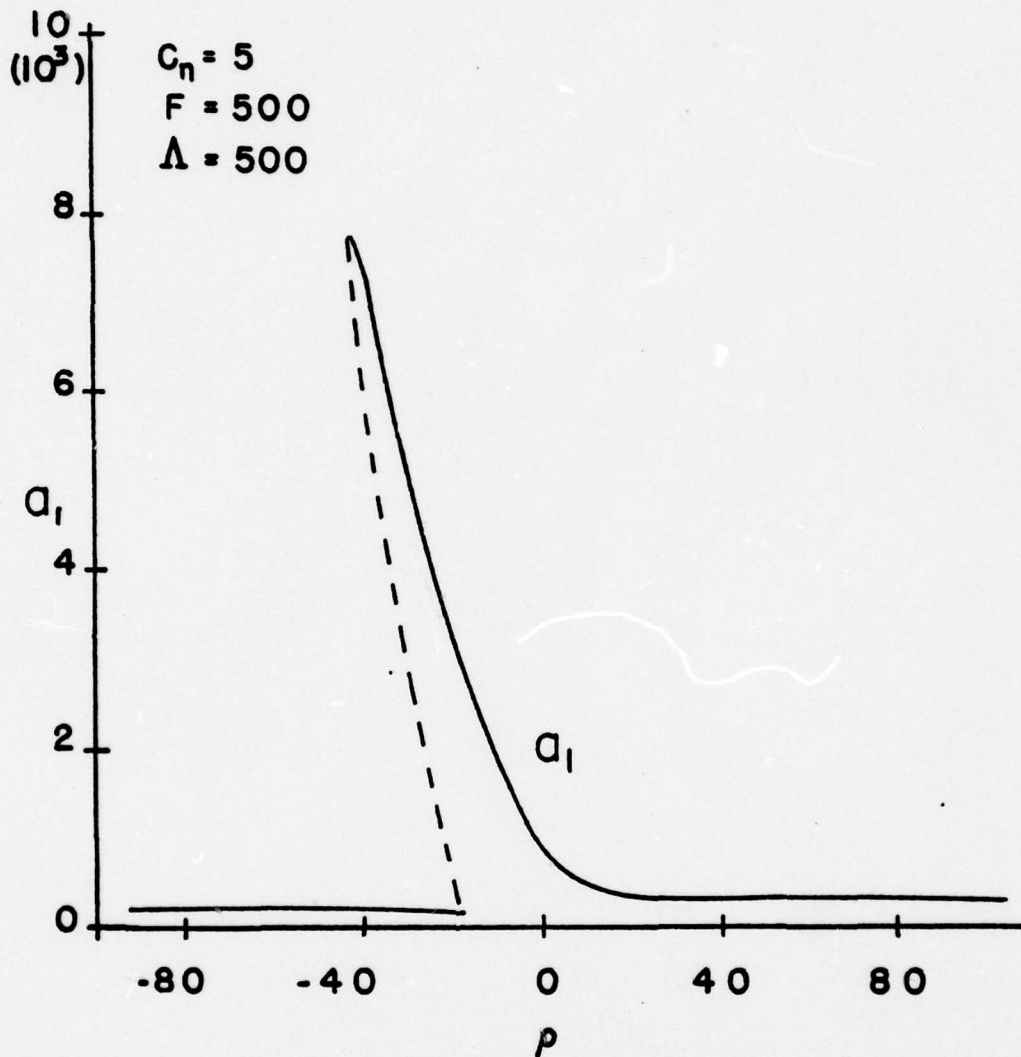


Figure 24. First modal amplitude as a function of excitation frequency,  $\rho(\epsilon\rho = \lambda - \omega_1 - \omega_3)$ , for constant amplitude of excitation,  $F$ , and constant aerodynamic detuning,  $\Lambda$ . — stable, ---- unstable.

A. Case (1) Where  $\lambda \approx 2\omega_1$ .

Equations (9.22) remain valid and the entire discussion, concerning the response of the first and second modes, of Section 9.3 is valid. The only change is that the third mode may now be drawn into the response through equations (7.6c,f) or

$$\omega_3 C_3 a_3 - \frac{1}{8} \Gamma_{3222} a_2^3 \sin \beta + \frac{f_{32} a_2}{2} \sin \gamma = 0 \quad (9.28a)$$

$$\omega_3 a_3 \left( \frac{3\rho}{2} - \sigma \right) + \frac{1}{8} \Gamma_{3222} a_2^3 \cos \beta - \frac{f_{32}}{2} a_2 \cos \gamma = 0 \quad (9.28b)$$

where for  $\epsilon = .001$

$$\sigma = \frac{22.066 - 3(8.1076)}{(.001)^{1/2}} = -71.36 \quad (9.28c)$$

Using a Newton-Raphson technique we find that the response is identical to that of Section 9.3 for no internal resonance. The relative sizes of  $a_1$  and  $a_2$  are controlled by the damping and detuning of the parametric resonance. The amplitude  $a_3$  is very small and the response appears as shown in Figures 16 - 21. To obtain a qualitative feel for the conditions under which  $a_3$  is drawn into the response, we let the amplitude  $F$  of the forcing term be small. Then equations (9.28) can be combined to yield

$$a_3 \approx \frac{\Gamma_{3222} a_2^3}{8\omega_3 \sqrt{C_3^2 + \left( \frac{3\rho}{2} - \sigma \right)^2}} \quad (9.29)$$

Thus,  $a_3$  will be small as long as the radical of (9.29) is large. For small damping the radical will be small only if

$$\frac{3\rho}{2} - \sigma \approx 0 \quad (9.30)$$

Since the internal-resonance detuning,  $\sigma$ , is fixed by virtue of the form of the boundary conditions, then the parametric detuning must be very nearly  $2\sigma/3$  for  $a_3$  to be significant. However, in this case  $\sigma$  is large and thus  $\rho$  must also be large. But from Section 9.3 we note that a large detuning caused  $a_1$  to be so large that  $a_2$  was negligible by comparison. So that for large detuning of both internal and parametric resonances the result is essentially that of Figures 16, 17, and 18.

As previously discussed, the detuning of the internal resonance is governed by the boundary conditions. For small damping and small detuning the response is depicted essentially as described by Figures 19 and 20 with the addition of the third mode; see Figures 25 and 26. Thus any of the three modes can be made greater than the other two, depending on the type of boundary conditions, damping, and detuning of the parametric excitation. A very striking representation of the relative sizes of the modes is portrayed in Figures 27 and 28. We note here that the response of the third mode is the same as that for a linear oscillator with a single degree-of-freedom. The second mode amplitude  $a_2$  merely acts as the external forcing function. A comparison with Figure 18 can be made and we see how the decrease in damping allows the second and third modes to enter the response. The second mode can be seen to be larger than the first when the detuning of the excitation is near zero. Detuning the internal resonance shifts the response of the third mode considerably.

#### B. The Case of No Parametric Resonance.

This is essentially a special case of the previous one. In the absence of a parametric resonance the response for the first two modes



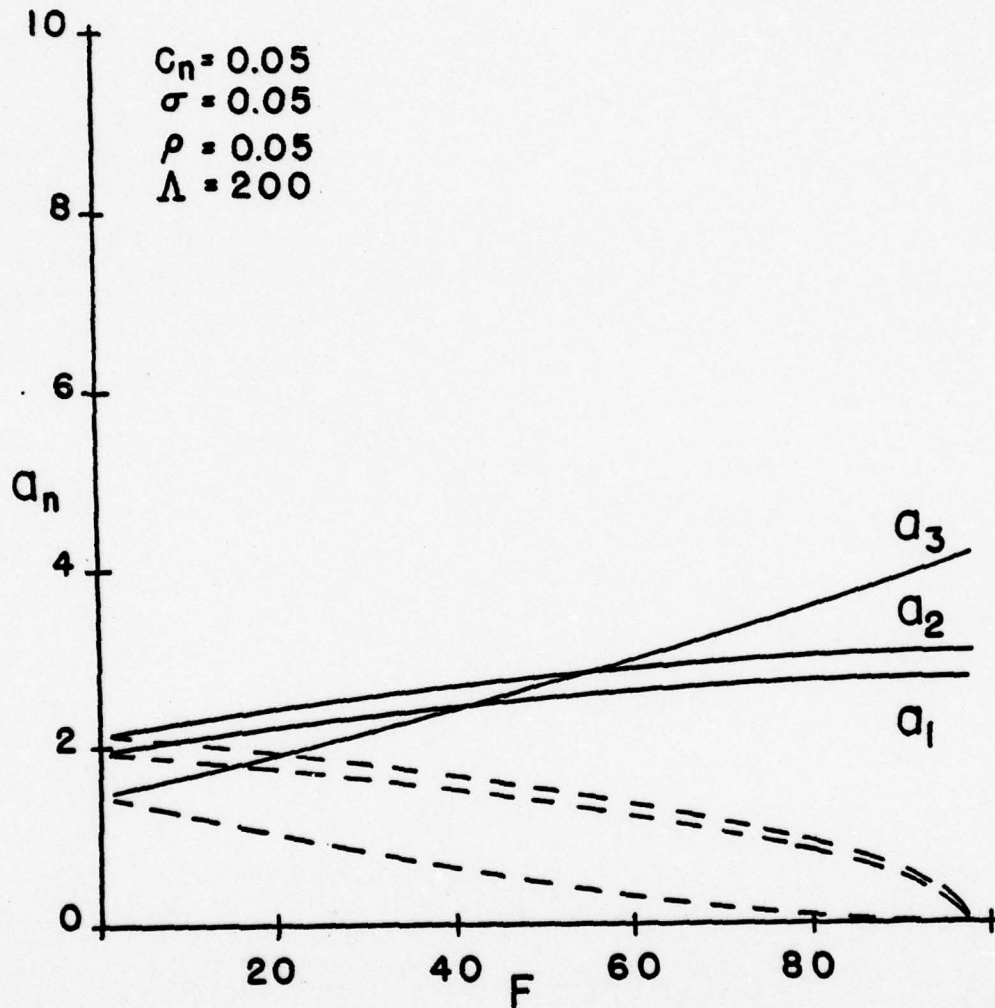


Figure 25. The modal amplitudes as functions of the amplitude of excitation,  $F$ , for constant frequency of excitation,  $\rho(\epsilon\rho = \lambda - 2\omega_1)$ , constant aerodynamic detuning,  $\Lambda$ , and internal resonance,  $\sigma(\epsilon\sigma = \omega_3 - 3\omega_1)$ .  
 — stable, ---- unstable.

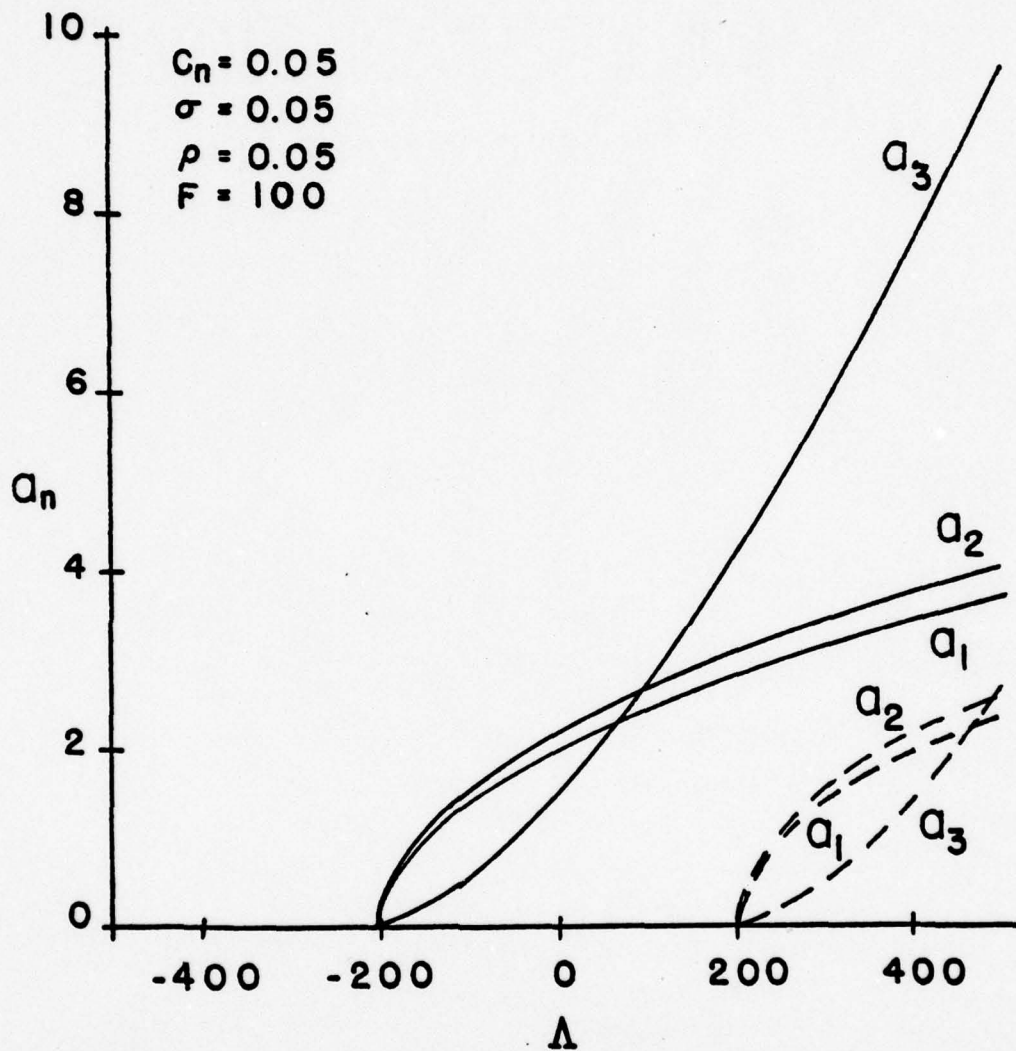


Figure 26. The modal amplitudes as functions of the aerodynamic detuning,  $\Lambda$ , for constant frequency of excitation,  $\sigma(\epsilon\sigma = \lambda - 2\omega_1)$ , constant amplitude of excitation,  $F$ , and internal resonance  $\sigma(\epsilon\sigma = \omega_3 - 3\omega_1)$ .  
 — stable, ---- unstable.

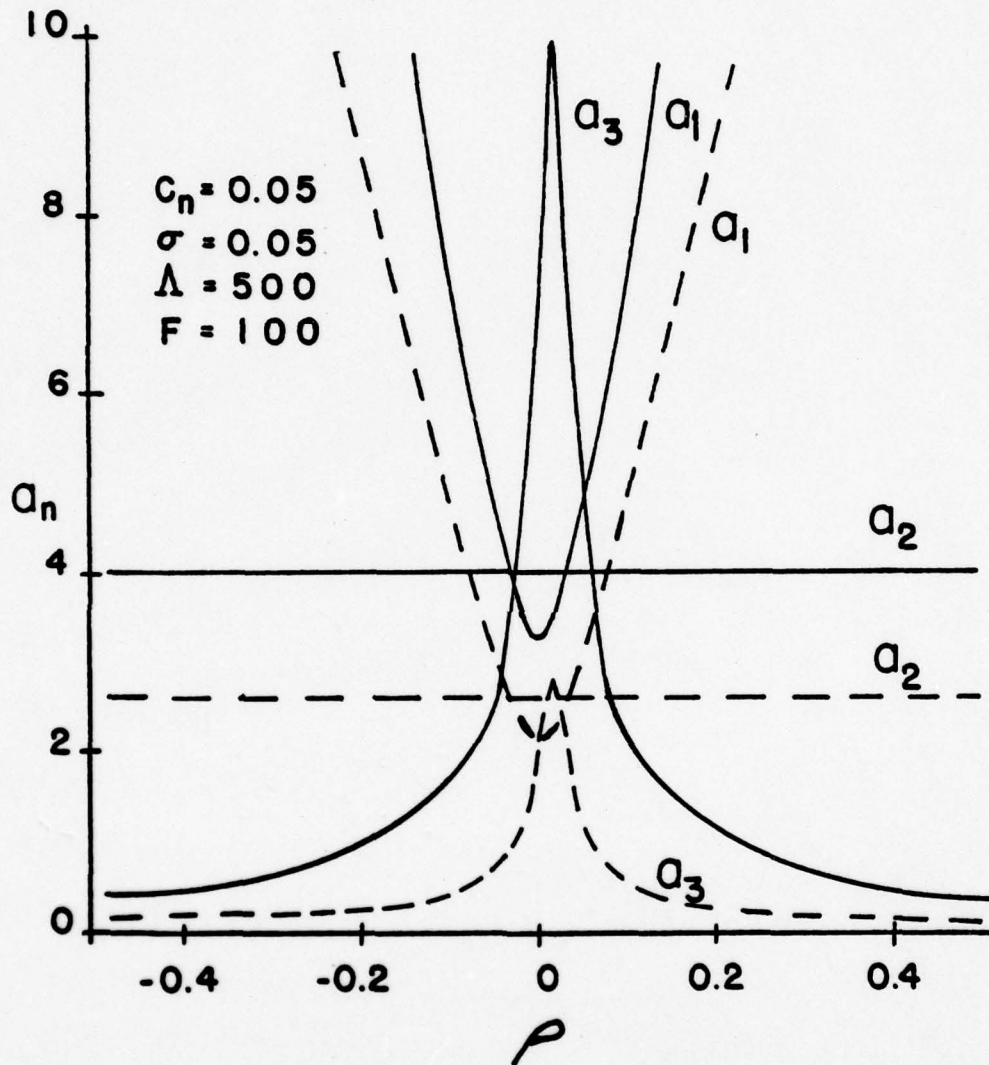


Figure 27. The modal amplitudes as functions of the frequency of excitation,  $\rho(\epsilon\rho = \lambda - 2\omega_1)$ , for constant amplitude of excitation,  $F$ , constant aerodynamic detuning,  $\Lambda$  and internal resonance  $\sigma(\epsilon\sigma = \omega_3 - 3\omega_1)$ . — stable, ---- unstable.

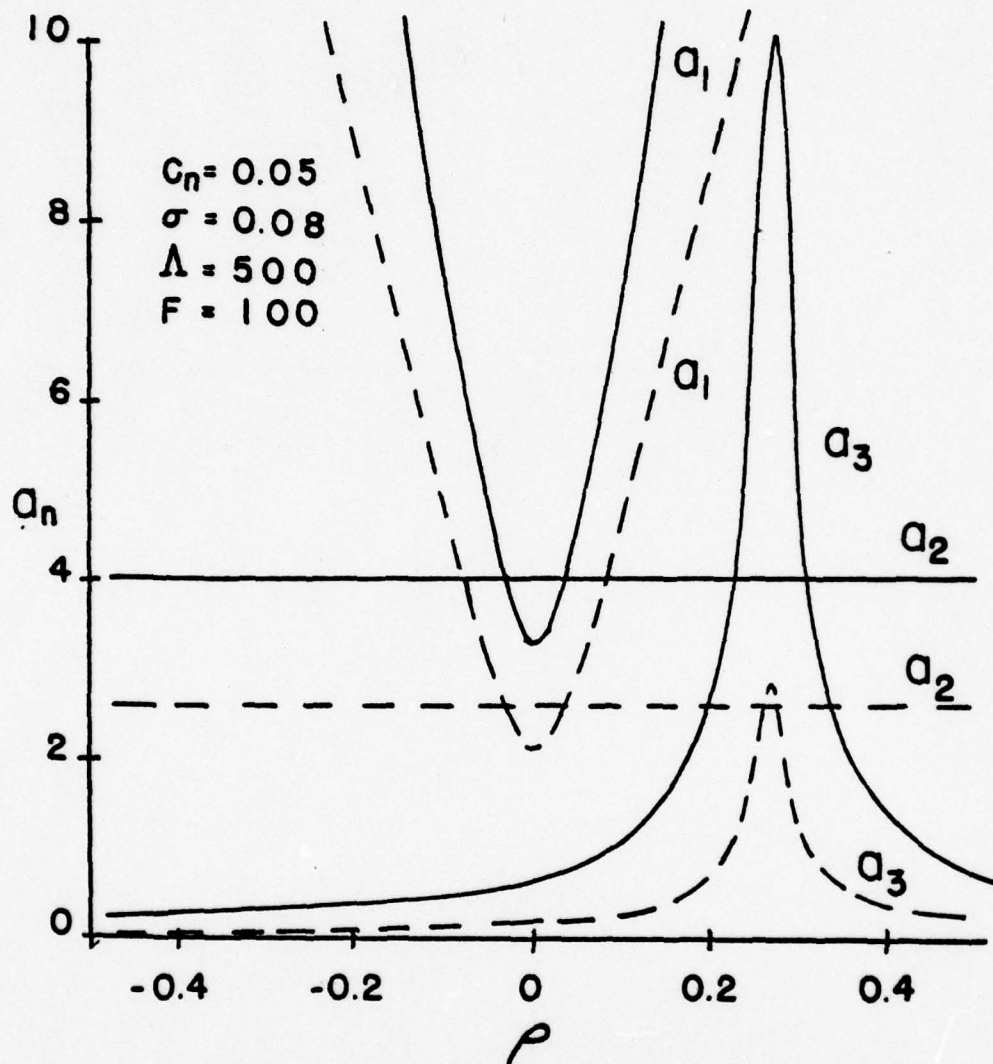


Figure 28. The modal amplitudes as functions of the frequency of excitation,  $\rho(\epsilon\rho = \lambda - 2\omega_1)$ , for constant amplitude of excitation,  $F$ , constant aerodynamic detuning,  $\Lambda$  and internal resonance  $\sigma(\epsilon\sigma = \omega_3 - 3\omega_1)$ . — stable, ---- unstable.



remains the same, or

$$a_2 = \sqrt{\frac{4\omega_1^2 C_1 C_2 - \Lambda \beta_{12}}{3/4 \Gamma_{1222}}} \quad (9.35a)$$

$$a_1 = 2\omega_1 C_2 a_2 \quad (9.35b)$$

and the third mode response is

$$a_3 = \frac{\frac{1}{8} \Gamma_{3222} a_2^3}{\sqrt{\omega_3^2 (C_3^2 + \sigma^2)}} \quad (9.35c)$$

The response is as shown in Figure 29. Again any of the three modes may be the largest depending on the damping and detuning of the internal resonance. For large damping  $a_1 > a_2$  or  $a_3$ . For small damping  $a_3$  or  $a_2 > a_1$  depending on the detuning of the internal resonance. In addition, the larger the dynamic pressure is the more likely the third mode will be greatest.

#### 9.5. Summary

The method of solution described in Chapters 5, 6, and 7 is used to solve numerical examples of a simply supported plate undergoing cylindrical bending. Various combinations of internal and parametric resonances have been examined.

When neither internal nor parametric resonances are present the response depends on the extent to which the dynamic pressure exceeds the critical value for inducing flutter. The amplitude of the response increases as the dynamic pressure increases. Both modes associated with the repeated frequency are excited. Only one stable solution exists and it is either trivial or nontrivial depending on the dynamic pressure.

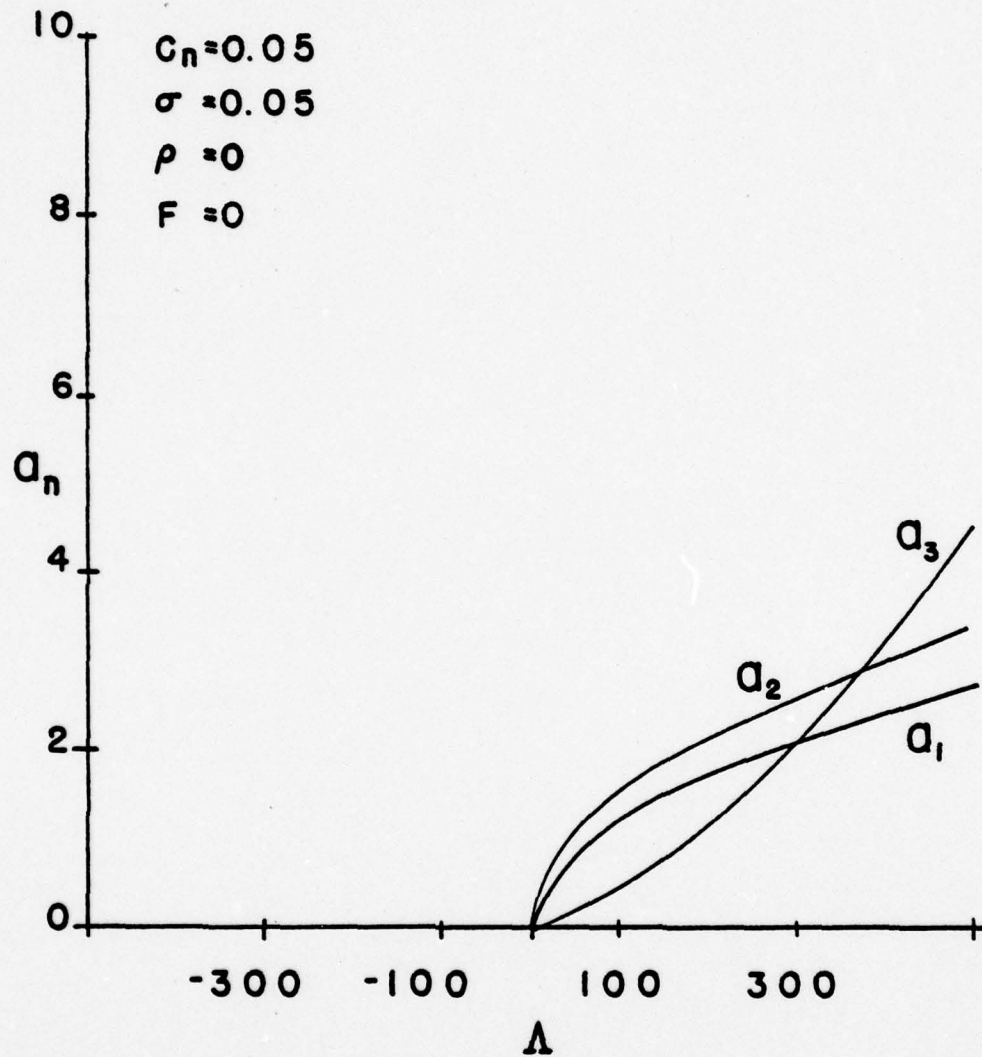


Figure 29. The modal amplitudes as functions of the aerodynamic detunings,  $\Lambda$ , for constant internal resonance  $\sigma(\epsilon\sigma = \omega_3 - 3\omega_1)$  and no parametric resonance.

When a parametric resonance of the form  $\lambda$  near  $2\omega_1$  or  $\omega_3 \pm \omega_1$  is present, then two nontrivial solutions can exist but only one is stable. The only modes excited are associated with the repeated frequency. Jump phenomena can be produced by varying the frequency or amplitude of the parametric excitation or the dynamic pressure.

When both internal and parametric resonances are present all three modes can be excited. The detunings of internal and parametric resonances have a great effect on the third and second modes, respectively. When the detuning of the internal resonance is large, as in the case of the simply supported plate considered, we see that the response is essentially identical to the case of no internal resonance. Jump phenomena exist regardless of which amplitude is largest.



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## APPENDIX A

### SIMILARITY TRANSFORMATIONS

#### A.1 General

Given a nonsingular matrix  $Z$ , then

$$[J] = [Z^{-1}][B][Z] \quad (A1)$$

is a similarity transformation where the matrices  $[J]$  and  $[B]$  have the same eigenvalues. The choice of  $[Z]$  can determine the complexity of the form of  $[J]$ , the simplest being the diagonal form when all the eigenvalues are distinct. If the eigenvalues are not all distinct  $[J]$  may still be diagonal, if the eigenvectors corresponding to the repeated roots are distinct. When the eigenvectors corresponding to repeated roots are identical, then the matrix can at best take on the Jordan form, where the eigenvalues appear as diagonal terms and a 1 (one) appears as an off-diagonal term adjacent to the repeated roots; see equation (9.11b). The number of off-diagonal terms depends on the number of repeated frequencies or more precisely the number of times a given root is repeated, and the corresponding number of independent eigenvectors. The latter depends on the rank of the matrix which is not the major topic of discussion of this appendix. However, a brief aside is in order. The reader must accept what follows in this example with faith or confirm it with rigor in a mathematics text such as Kreyszig (1972) (pp. 256-279).

Given an  $n \times n$  matrix with one repeated frequency, then, if the rank of the matrix is  $r = n - 1$ , there is only one linearly independent

eigenvector associated with the repeated frequency. If the rank is  $r = n - 2$  then there are two linearly independent eigenvectors associated with the repeated frequency. The last case will allow generation of a diagonal matrix while the first will produce the Jordan form with one off diagonal term.

The problem is one of determining the matrix  $[Z]$  which will put  $[B]$  in Jordan form. Mathematics texts indicate  $[Z]$  exists and give examples to prove it or direct the student to verify a sample problem with mechanical computations. However, nowhere, does there appear to be a method to determine the matrix  $[Z]$  in the first place. That is precisely the purpose of this Appendix.

Finding the eigenvalues and eigenvectors of  $[B]$  is a relatively trivial problem with the advent of sophisticated library computer programs. Using that as a starting point we proceed with a Newton-Raphson iterative technique to generate the matrix  $[Z]$  from the set of eigenvectors.

#### A.2 Determining the Form of $[J]$ .

We start with the matrix  $[B]$  and use any standard library computer program which will return eigenvalues and eigenvectors. An inspection of eigenvalues will indicate immediately if any are repeated. If they are too numerous, a simple sort routine can be employed to order them from highest to lowest to aid in identifying repeated roots. Once the repeated roots are identified, the corresponding eigenvectors are compared. The best way is to normalize them by dividing the elements, usually returned as a column vector, by the value of the first element.

The vectors which remain are either identical or they are not. If they are identical then the Jordan form will evolve. There need not be any discussion of rank, linear independence or whatever.

### A.3 Newton-Raphson Iteration Technique

The Newton-Raphson technique is especially useful in solving nonlinear problems. It is "initial guess" sensitive to some degree but converges rapidly if the system is fairly well behaved near the solution. The basic problem is one where we are required to find the values of a vector  $\{x\}$  which will make

$$\{f(x)\} = 0 \quad (A2)$$

For a 2 x 2 example we expand in a Taylor series and obtain

$$f_1(x_1) = f_1(x_{10}) + \frac{\partial f_1}{\partial x_1}(x_{10})\Delta x_1 + \frac{\partial f_1}{\partial x_2}(x_{10})\Delta x_2 + \dots \quad (A3a)$$

$$f_2(x_2) = f_2(x_{20}) + \frac{\partial f_2}{\partial x_1}(x_{20})\Delta x_1 + \frac{\partial f_2}{\partial x_2}(x_{20})\Delta x_2 + \dots \quad (A3b)$$

where the  $x_{n0}$  are the initial guesses. The  $\Delta x_n$  are the increments which will cause all the  $f_i(x_n)$  to be zero simultaneously, so we write

$$\frac{\partial f_1}{\partial x_1} \Delta x_1 + \frac{\partial f_1}{\partial x_2} \Delta x_2 = -f_1(x_{10}) \quad (A4a)$$

$$\frac{\partial f_2}{\partial x_1} \Delta x_1 + \frac{\partial f_2}{\partial x_2} \Delta x_2 = -f_2(x_{20}) \quad (A4b)$$

and we have a linear system which can be solved for the  $\Delta x_n$  to add to the  $x_{n0}$  and proceed to another iteration until we reach some pre-determined limit of convergence in either the  $f_i$  or  $\Delta x_i$ .



#### A.4 The [Z] Matrix With One Repeated Frequency

The [Z] matrix is constructed as follows. The eigenvectors are sorted so that the repeated vectors are in the first two columns and are normalized so the top element in each column is one. We seek a column vector to replace the first column, the repeated eigenvector. The vector function,  $f$ , represents the first column of the matrix [J] below the first element; that is

$$\{J_1\} = \begin{bmatrix} 1 \\ f_1(x_1) \\ f_2(x_2) \\ \cdot \\ \cdot \\ f_{n-1}(x_{n-1}) \end{bmatrix} \quad (A5a)$$

We wish the final result to be

$$\{J_1\} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad (A5b)$$

If the functional form  $f(x_n)$  were known, we would have no problem in evaluating the partial derivatives

$$\frac{\partial f_i}{\partial x_j} \quad (A6)$$

Here we must construct them numerically. We begin by assuming the first column of the  $[Z]$  matrix to be

$$\{Z_1\}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad (A7)$$

Then we compute

$$[J] = [Z^{-1}][B][Z] \quad (A8)$$

The first column of  $[J]$  becomes

$$\{J_1\}_0 = \begin{bmatrix} 1 \\ f_1(x_0) \\ f_2(x_0) \\ \cdot \\ \cdot \\ f_{n-1}(x_0) \end{bmatrix} \quad (A9)$$

Then we set

$$\{Z_1\}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad (A10)$$

and compute (A8) again so that

$$\{J_1\}_1 = \begin{bmatrix} 1 \\ f_1(x_1) \\ f_2(x_2) \\ f_3(x_3) \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \quad (A11)$$

We may now compute the derivatives numerically as

$$\frac{\partial f_1}{\partial x_1} = \frac{f_1(x_1) - f_1(x_0)}{\Delta x_1} = f_1(x_1) - f_1(x_0) = f_{11} \quad (A12a)$$

$$\frac{\partial f_2}{\partial x_1} = \frac{f_2(x_1) - f_2(x_0)}{\Delta x_1} = f_2(x_1) - f_2(x_0) = f_{21} \quad (A12b)$$

Since  $\Delta x_1 = 1$ . We repeat for

$$\{Z_1\}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad (A13)$$

to obtain

$$\{J_1\}_2 = \begin{bmatrix} 1 \\ f_1(x_2) \\ f_2(x_2) \\ \cdot \\ \cdot \\ f_{n-1}(x_2) \end{bmatrix} \quad (A14)$$

so that the second column of derivatives may be computed according to

$$\frac{\partial f_1}{\partial x_2} = \frac{f_1(x_2) - f_1(x_0)}{\Delta x_2} = f_1(x_2) - f_1(x_0) = f_{12}$$



and so on, until we have a system of the form of (A4) and we write for

$$r = n - 1$$

$$\begin{bmatrix} f_{11} & f_{12} & . & . & . & f_{1n} \\ f_{21} & . & . & . & . & . \\ f_{31} & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ f_{n1} & . & . & . & . & f_{nn} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ . \\ . \\ . \\ \Delta x_n \end{bmatrix} = \begin{bmatrix} 1 - f_1(x_1) \\ - f_2(x_2) \\ - f_3(x_3) \\ . \\ . \\ - f_n(x_n) \end{bmatrix} \quad (\text{A15})$$

where the right side was computed from

$$\{Z_1\}_R = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad (\text{A16})$$

and the factor of one applied to the first element to satisfy (A5b).

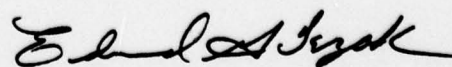
Equations (A15) may now be solved for the  $\Delta x_i$  and added to one to form a new guess for the  $x_i$ . Now the product of (A1) is computed to determine if the Jordan form has been achieved. Convergence is achieved by specifying the accuracy with which (A17) is satisfied:

$$[Z^{-1}][B][Z] - [J] \approx 0 \quad (\text{A17})$$

In this problem one step was required to produce an accuracy of  $10^{-10}$  in the element using double precision.

## VITA

The author was born in Steelton, Pennsylvania, on 16 October 1940. He graduated from Central Darphin High School in Harrisburg, PA in 1958. He attended the U. S. Military Academy at West Point, NY and graduated in 1963 as a Distinguished Cadet (top 5% of the class) and was commissioned as a Second Lieutenant in the U. S. Army. Between 1963 and 1965 he attended the Engineer Officers Basic Course at Fort Belvoir, VA, Airborne Training at Fort Benning, GA, and served as Executive Officer and Company Commander of B. Co. 13th Engineer Battalion, 7th Infantry Division, Korea. In 1967 he received a Master of Science degree in Astrodynamics from the University of California at Los Angeles (UCLA). He then spent a year as an Engineer Construction Battalion Advisor in Vietnam and followed that with attendance at the Amphibious Warfare School in Quantico, VA. Between 1969 and 1972 he spent a year as an Instructor for Engineering Mechanics and two years as an Assistant Professor for Space Mechanics in the Mechanics Department at the U. S. Military Academy. This was followed by a second Vietnam tour where he served as the Plans Officer for the U. S. Army Engineer Group. He is currently the Group Director for solid mechanics courses for the Department of Mechanics at the U. S. Military Academy.



PARAMETRICALLY EXCITED NONLINEAR  
MULTI-DEGREE-OF-FREEDOM SYSTEMS

by

Edward G. Tezak

(ABSTRACT)

An analysis of parametrically excited nonlinear multi-degree-of-freedom systems is presented. The nonlinearity considered is cubic and small so that the system of equations is weakly nonlinear. Modal damping is included and the parametric excitation is harmonic. The systems examined include those with distinct natural frequencies as well as those with a single repeated frequency. The significant role played by the existence of an internal resonance is explored in depth. The derivative-expansion version of the method of multiple scales, a perturbation technique, is used to develop solvability conditions for the various combinations of internal and parametric resonances considered.

Regions where trivial and nontrivial solutions exist are defined and the stability of the solutions within each region is discussed. Nontrivial, unstable solutions have been shown to exist in regions where nontrivial stable solutions are known. Numerical solutions do not hint at the existence of these solutions.

The role of internal resonance in parametrically excited systems is explored. Strong modal interaction is demonstrated as a consequence of the presence of the cubic nonlinearity and the internal resonance.



Because of this modal coupling, modes other than the one excited can dominate the response. A multiplicity of jumps is shown to exist.

Parametric excitation in the nonlinear flutter problem has been examined in detail for the first time. The effect of the parametric excitation can raise the flutter speed if the amplitude is small and lower it if the amplitude is large.

Limit-cycle behavior in the flutter problem is developed in a unique analytic way. The condition which predicts the onset of flutter is developed by using a linear analysis. The solution grows without bound. Interestingly, the same condition in the nonlinear analysis predicts the existence of a real nontrivial solution with a finite amplitude, the so-called limit-cycle.